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Metrical Problems in Minkowski Geometry

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Abstract

In this dissertation we study basic metrical properties of 2-dimensional normed linear spaces, so-called (Minkowski or) normed planes.

In the first chapter we introduce a notion of angular measure, and we investigate under what conditions certain angular measures in a Minkowski plane exist. We show that only the Euclidean angular measure has the property that in an isosceles triangle the base angles are of equal size. However, angular measures with the property that the angle between orthogonal vectors has a value of $\frac{\pi}{2}$, i.e, a quarter of the full circle, exist in a wider variety of normed planes, depending on the type of orthogonality. Due to this we have a closer look at isosceles and Birkhoff orthogonality. Finally, we present results concerning angular bisectors.

In the second chapter we pay attention to convex quadrilaterals. We give definitions of different types of rectangles and rhombi and analyse under what conditions they coincide. Combinations of defining properties of rectangles and rhombi will yield squares, and we will see that any two types of squares are equal if and only if the plane is Euclidean. Additionally, we define a "new" type of quadrilaterals, the so-called codises. Since codises and rectangles coincide in Radon planes, we will explain why it makes sense to distinguish these two notions. For this purpose we introduce the concept of associated parallelograms.

Finally we will deal with metrically defined conics, i.e., with analogues of conic sections in normed planes. We define metric ellipses (hyperbolas) as loci of points that have constant sum (difference) of distances to two given points, the so-called foci. Also we define metric parabolas as loci of points whose distance to a given point equals the distance to a fixed line. We present connections between the shape of the unit ball and the shape of conics. More precisely, we will see that straight segments and corner points of B cause, under certain conditions, that conics have straight segments and corner points, too. Afterwards we consider intersecting ellipses and hyperbolas with identical foci. We prove that in special Minkowski planes, namely in the subfamily of polygonal planes, confocal ellipses and hyperbolas intersect in a way called Birkhoff orthogonal, whenever the respective ellipse is large enough.

Key Words: angular measure, Birkhoff orthogonality, codis, conic section, isosceles orthogonality, metric ellipse, metric hyperbola, metric parabola, metric parallelogram, Minkowski plane, normed plane, parallelogram, Radon plane, rectangle, rhombus, smooth plane, square, strictly convex plane

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Contents

1 Introduction

1.1 Motivation and some history

In Euclidean Geometry, "length" is uniform and independent of direction. This means that the locus of points that have equal distance from a given point is the usual circle, a perfectly round and highly symmetric object. Unfortunately, sometimes this model cannot be applied in the "real world". The length of a tour for example is rather measured in "the time it takes to get from A to B" than in the pure distance.

In view of this it seems to be reasonable to alter the way in which distances are measured. (For example, one kilometre uphill is larger than one kilometre downhill in the sense that it takes more time.) As a consequence, the "circle" changes its shape into some other curve.

Minkowski Geometry adopts this idea under some conditions that are not necessarily fulfilled in the real world:

i) the shape of the unit circle (circle with radius 1) is independent of the location of the circle,
ii) the unit circle is symmetric, and
iii) the unit circle is convex.

As a conclusion, Minkowski Geometry is the geometry of real finite-dimensional normed spaces. It is a generalisation of Euclidean Geometry, and a special case of *Finsler Geometry* [17] (which is achieved if condition i) is omitted). Condition ii) means that only the direction is important for measuring, not the orientation. Leaving it out, we get the *geometry of gauges*. Finally, iii) is equivalent to the fact that the triangle inequality holds.

Minkowski Geometry is named after Hermann Minkowski, who introduced the axioms of Minkowski spaces in 1896 [32]; a recent important monograph on Minkowski Geometry is [40].

The first reference to geometry in the sense of Minkowski seems to be given in the famous "Habilitationsvortrag" of Riemann [35], who dealt with the l_4-norm, roughly thirty years before Minkowski. Early and important works on the topic are due to Gołab [20, 19] and Busemann [8]. Minkowski Geometry plays a role in several mathematical disciplines, such as Combinatorics, Discrete Geometry, Functional Analysis and Optimisation.

Our approach to the topic will be mainly geometric in nature, which means that we try to avoid coordinates (and everything based on them, like slopes of lines etc.) whenever it is possible. Instead of this, we often try to get a clear *descriptive picture* of what is going on, and we support our ideas by the frequent use of appropriate figures.

1.2 Preliminaries

1.2.1 Notation

As described above, Minkowski Geometry is the geometry of normed spaces, i.e., of finite dimensional normed Banach spaces. Thus it is reasonable to start by explaining what we mean with "normed".

Definition 1 *Let X be a real linear vector space with origin o. A* norm *is a function $||\cdot|| : X \to \mathbb{R}$ that satisfies the following properties:*

i) $||x|| \geq 0$ *for all $x \in X$,*
ii) $||x|| = 0 \iff x = o$,
iii) $||ax|| = |a| \cdot ||x||$ *for all $a \in \mathbb{R},\ x \in X$,*
iv) $||x + y|| \leq ||x|| + ||y||$ *for all $x, y \in X$.*

By $(X, ||\cdot||)$ we denote the space X endowed with the norm $||\cdot||$. If X is finite dimensional, $(X, ||\cdot||)$ is then called a (*normed* or) *Minkowski space*.

The geometry of X is fully determined by the set $B := \{x : ||x|| \leq 1\}$, that we call the *unit ball* of X. Its boundary $S := \{x : ||x|| = 1\}$ is called the *unit sphere* of X.

It is easy to see that the norm axioms imply that B is a convex body (i.e., a bounded, closed, convex set with interior points) that is symmetric to o. This can also be said as follows: B is a *centered* convex body in $\mathbb{R}^d$ endowed with the norm $||\cdot||_B$ (which is defined in the following proposition).

Also the opposite is true; see, e.g., Thompson [40].

Proposition 1 *Let $B \subseteq X$ be a convex body with non-empty interior and symmetric to o. Then B defines a norm on X by*

$$||x||_B := \inf\{\lambda > 0 : x \in \lambda \cdot B\} \quad \text{for all } x \in X.$$

We are going to use this more geometric approach rather than the analytic one that defines the norm directly. To aid this point of view, we denote a Minkowski space by $M^d(B)$, where $d < \infty$ is the dimension of X. We will simply write $||\cdot||$ instead of $||\cdot||_B$ if it is clear that the norm is defined by B.

In the case $d = 2$ we call $M^d(B)$ a *Minkowski plane* with *unit disc B* and *unit circle S*. If B is an ellipsoid, then $M^d(B)$ is isometric to the d-dimensional Euclidean space, and we call $M^d(B)$ *Euclidean*.

If x and y are two different points of the space, we write

$$[x, y] := \mathrm{conv}\{x, y\} = \{\lambda x + (1 - \lambda)y,\ \lambda \in [0, 1]\}$$

for the convex hull of x and y and call it the *(straight) segment* joining x and y. By

$$(x, y) := \mathrm{aff}\{x, y\} = \{\lambda x + (1 - \lambda)y,\ \lambda \in \mathbb{R}\}$$

we denote the affine hull of x and y, and we call it the *line* passing through x and y.

If $x, y, z \in M^d(B)$ are in convex position (i.e., none of the three points is contained in the convex hull of the remaining two points), we write

$$\Delta xyz := \mathrm{conv}\{x, y, z\}$$

for the *triangle* with *vertices* (or *corner points*) $x,\ y$ and z.

1.2.2 Strictly convex and smooth Minkowski planes

Let the line L divide the plane into two (closed) half-planes, and let C be a curve, i.e., the image of the interval $[0, 1] \subseteq \mathbb{R}$ under any continuous function. Then L is said to *support* C at $x \in C$, or to be a *supporting line* of C at x, if $x \in L$ and there is a neighbourhood $N(x)$ of x such that $N(x) \cap C$ is entirely contained in one of the two half-planes.

The sets B and S, as well as the plane $M^2(B)$, are called *smooth* if for each point $p \in S$ there is a unique line supporting the (closed) curve S at p. They are called *strictly convex*, if S does not contain any straight line segment. See Figure 1 for examples.

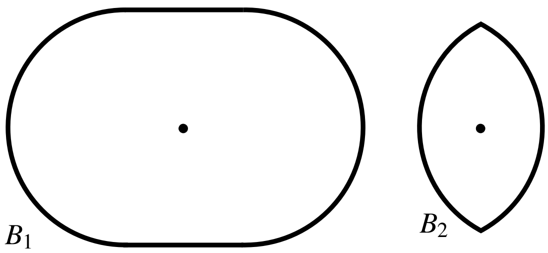

Figure 1 – B_1 is not strictly convex, but smooth. B_2 is not smooth, but strictly convex.

1.2.3 Orthogonality

In Minkowski Geometry, there is no "natural" concept of orthogonality. Thus, huge parts of the classic elementary geometry (Pythagoras' theorem, Theorem of Thales and so on) do no longer hold. Similarly, the topic of area measuring ("height times baselength") has to be considered in a new way.

There are several possibilities to define two vectors to be *normal* (or *orthogonal*) to each other, all of which are equivalent in the Euclidean plane. We work with two of them.

An important orthogonality type is the *Birkhoff orthogonality*, and it was introduced by Birkhoff [5].

Two vectors $x, y \in M^2(B)$ are said to be *orthogonal in the sense of Birkhoff*, or *Birkhoff orthogonal*, abbreviated by $x \perp_B y$, if $||x|| \leq ||x + t \cdot y||$ for all $t \in \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Geometrically this means that there is a line parallel to y that supports the unit ball B at $\frac{x}{||x||}$ (unless $x = o$ or $y = o$).

Birkhoff orthogonality is homogeneous, i.e., $x \perp_B y \Rightarrow x \perp_B \alpha y$ for $\alpha \in \mathbb{R}$. But in general it is not a symmetric relation; see Figure 2 for an example. If for all $x, y \in M^2(B)\backslash\{o\}$ (or for all $x, y \in S$, which is obviously equivalent) we have that $x \perp_B y \Rightarrow y \perp_B x$, then $M^2(B)$ is said to be a *Radon plane*; the concept of *Radon curves* (the unit circles of Radon planes) was first introduced by Radon [34]. These "Euclidean-like" curves are well studied; see [31] for further information. We remark that (regarding Birkhoff orthogonality) there is no higher dimensional analogue to Radon curves: Whenever in a Minkowski space with dimension at least 3 we have that Birkhoff orthogonality is symmetric, then the space is Euclidean [6].

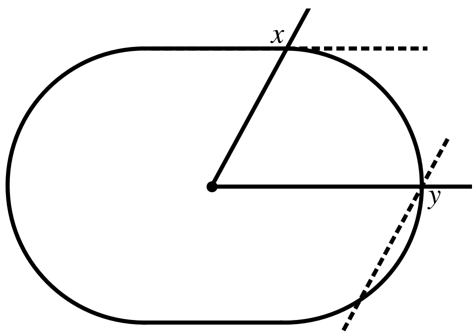

Figure 2 – We have $x \perp_B y$, but not $y \perp_B x$.

Another type of orthogonality that will be discussed in this dissertation is the *James* or *isosceles orthogonality*, introduced by James [23].

We call two vectors $x, y \in M^2(B)$ *James orthogonal* or *isosceles orthogonal*, denoted by $x \perp_I y$, if $||x + y|| = ||x - y||$. Geometrically this means that the triangle with vertices x, $-x$, and y is isosceles.

James orthogonality is symmetric, and it is homogeneous if and only if the plane under consideration is the Euclidean one [23].

There are several characterizations of the Euclidean space based on these two notions. For us, the following ones are especially important:

- $x \perp_I y \Rightarrow x \perp_B y \quad \forall x, y \in M^2(B)$ (see [40], p. 87, or [3], (4.1)),
- $x \perp_B y \Rightarrow x \perp_I y \quad \forall x, y \in M^2(B)$ (see [3], (4.4)),
- $(u + v) \perp_B (u - v) \quad \forall u, v \in S$ (see [3], (4.2)), and
- $x \perp_B y \Rightarrow x \perp_I y \quad \forall x, y \in S$ (only in Radon planes; see [3], (10.2)).

For information on other types of orthogonality, as well as for relations between them, we refer to [1, 2]. More characterisations of the Euclidean space in terms of orthogonality types can be found in [3].

2 On angular measures

2.1 Introduction

It is well known that measuring angles is a very old topic in mathematics, and not only there. It is important for architecture, navigation, warfare (ballistic) and so on. In Euclidean Geometry, it is very natural to divide the circle into equal parts, and to declare angles of equal size (in this sense) to be equal.

In Minkowski Geometry it is not clear at first glance, what "equal size" of angles means. “Size” can refer to "arc length", measured in the norm, or to the area of the corresponding sector of the (unit) disc. Alternatively, one can look for equal angles in Euclidean Geometry and declare that angles satisfying analogous properties in normed planes have “equal size”; for example, base angles in isosceles or equilateral triangles, or angles between orthogonal lines. There are several possibilities to extend the notions of “angular measure” and “angles of equal size” to normed planes, and we will deal with some of them.

More precisely, we first define "angular measure" as a normed, symmetric measure on the unit circle S. Then we assume that the measure has a certain property, like the ones mentioned above, and study the consequences for the type of normed plane, i.e., under which conditions such an angular measure exists, and how it can be constructed.

The main results of this chapter were published by the author in [13, 14].

2.2 Definition and general properties

The notion of angular measure in Minkowski planes, as introduced by Brass [7], is fixed by the following

Definition 2 *Let μ be a measure on the unit circle S. The measure μ is called an* angular measure *if the following properties are satisfied:*

i) $\mu(S) = 2\pi$,
ii) $\mu(A) = \mu(-A)$ *whenever A is a measurable subsets of S, and where* $-A = \{u \in S \mid -u \in A\}$,
iii) $\mu(\{p\}) = 0$ *for any point $p \in S$,*
iv) *μ is translation invariant.*

Remark: Since here we have clearly a measure, an angular measure has the property of countable additivity, i.e., the measure of the union of countably many disjoint subsets of S is equal to the sum of the measures of these subsets.

Definition 3 *For $u, v \in S$ we define the* angle *between u and v as*

$$\sphericalangle(u,v) := \sphericalangle(u,o,v) := \mu(\widehat{uv}),$$

where $\widehat{uv}$ denotes the (small) arc from u to v. Further on, for points $a, b, c \in M^2(B)$, $a \neq b, b \neq c$, we define

$$\sphericalangle(a,b,c) := \sphericalangle\left(\frac{a-b}{||a-b||}, \frac{c-b}{||c-b||}\right).$$

Since by definition angular measures in normed planes are translation invariant, the angle $\sphericalangle(a,b,c)$ in Figure 3 has the same angular measure as the angle $\sphericalangle(\tilde{a}, \tilde{b})$, where $\tilde{a} = \frac{a-b}{||a-b||}$ and $\tilde{c} = \frac{c-b}{||c-b||}$.

In some situations, by "angle $\sphericalangle(a,b,c)$" we also refer to the geometric object that consists of the *apex* b and the two rays starting in b and passing through a and c. It is always clear from the context what we mean.

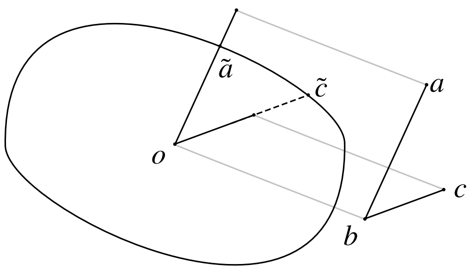

Figure 3 – Angular meaures are translation invariant.

The angle between $u, v \in S$ is called a *zero angle* if $\sphericalangle(u, v) = 0$, and it is called a *straight angle* if $\sphericalangle(u, v) = \pi$.

<u>Remark:</u> As long as there are no non-trivial zero angles, i.e., $\sphericalangle(u, v) > 0$ whenever $u \neq v$, all straight angles are of the form $\sphericalangle(u, -u)$, where $u \neq o$.

Since every angular measure μ is translation invariant, the sum of the interior angles of a triangle equals π. This can be proved in the same way as in the Euclidean case; see for example [24].

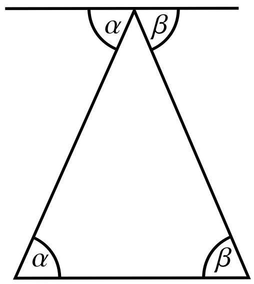

Figure 4 – Like in the Euclidean plane, in Minkowski planes the values of interior angles of a triangle sum up to π.

The angles denoted by α are equal due to properties $iv)$ and $ii)$ (following the Euclidean term we call them *alternate angles*); the same argument yields the equality of the angles β. Thus, the three interior angles of the triangle form a straight angle and, hence, sum up to π.

Lemma 1 *The following statements are equivalent:*

i) $\sphericalangle(-u, v, u) = \frac{\pi}{2} \quad \forall u, v \in S, u \neq v.$
ii) $\sphericalangle(x, y) = \frac{\pi}{2} \quad \forall x, y \in M^2(B)$ *with* $x \perp_I y$.

Proof: Assume that $i)$ holds, and let $x \perp_I y$. For $u = x - y, v = x + y$ we have

$$||u|| = || - u|| = ||v||,$$

and thus $\sphericalangle(x, y) = \sphericalangle(-u, v, u) = \frac{\pi}{2}$.

Let now $u, v \in S, u \neq v$. Since $||(u - v) + (u + v)|| = ||(u - v) - (u + v)||$, we have $(u - v) \perp_I (u + v)$, and thus $\sphericalangle(u - v, u + v) = \sphericalangle(-u, v, u) = \frac{\pi}{2}$. □

We remark that property $i)$ holds in the Euclidean plane and has been established most likely by Thales; see [12].

2.3 Motivation

Brass [7] considered angular measures with the property that in each equilateral triangle every interior angle has the value $\frac{\pi}{3}$. He proved the following

Theorem 1 *Let $M^2(B)$ be a Minkowski plane. There exists an angular measure μ such that every equilateral triangle is equiangular if and only if $M^2(B)$ is not the rectangular plane.*

Remark: A plane $M^2(B)$ is called *rectangular* if B is a parallelogram. Since these planes are affinely equivalent to each other, it makes sense to speak about *the* rectangular plane. It is the only normed plane with the property that its unit circle S contains a segment of length 2.

Proof of Theorem 1: Let $M^2(B)$ be the rectangular plane, and assume there exists such an angular measure. More precisely, let $B = \text{conv}\{a, b, -a, -b\}$, where $a, b \neq o$.

Consider the two triangles with vertices $o,\ a,\ \frac{1}{2}(a+b)$ and $o, \frac{1}{2}(a+b),\ b$, respectively. With the formula on the sum of interior angles we get that $\sphericalangle\left(a, o, \frac{1}{2}(a+b)\right)$ $= \sphericalangle\left(o, a, \frac{1}{2}(a+b)\right) = \sphericalangle\left(\frac{1}{2}(a+b), o, b\right) = \sphericalangle\left(o, b, \frac{1}{2}(a+b)\right) = \frac{\pi}{3}$; see Figure 5. Thus, the interior angles of the triangle with vertices o, a and b sum up to $\frac{4\pi}{3}$, a contradiction. Alternatively, we can consider the angles at the point $\frac{1}{2}(a+b)$, where the straight angle $\sphericalangle\,(a, \frac{1}{2}(a+b), b)$ has value $\frac{2\pi}{3}$.

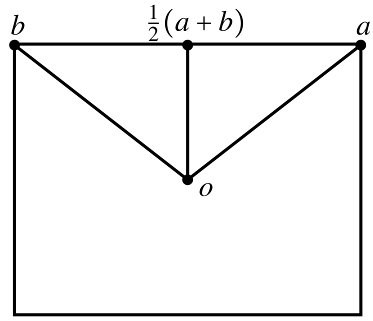

Figure 5 – Contradiction in the rectangular plane.

Let, on the other hand, $M^2(B)$ be a Minkowski plane with unit disc B such that the longest line segment in its boundary has length at most one, and hence for every $x \in S$ there is exactly one regular hexagon inscribed in S having x as vertex. Figure 6 shows a counterexample: The straight segments on the top and at the bottom (the segments that do not contain x or $-x$) have length greater than one. Thus, if H is a regular inscribed hexagon, its top and bottom segments can be shifted (independently) left and right without losing the required properties.

Let now $\varphi : \mathbb{R} \to S$ be a parametrisation of the unit circle such that

$$\sphericalangle\,(\varphi(s), \varphi(t)) = s - t \quad \forall t \leq s \leq t + \pi,$$

and let $h : \mathbb{R} \to \mathbb{R}$ be a continuous, monotonously increasing function such that

$$t < h(t) < t + \pi \text{ and } ||\varphi(h(t)) - \varphi(t)|| = 1 \quad \forall t \in \mathbb{R}.$$

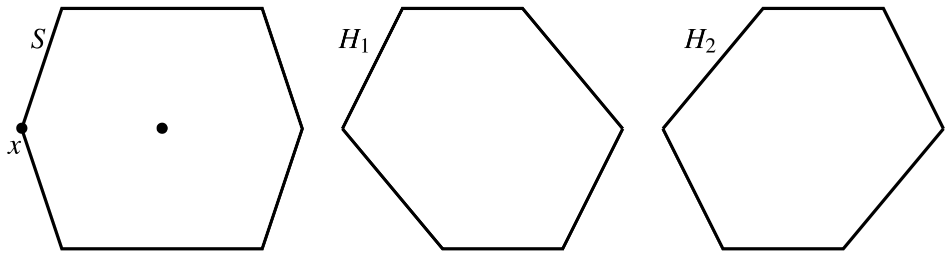

Figure 6 – H_1 and H_2 are (different) hexagons inscribed in S with cornerpoint x.

Then every triangle $\Delta o\varphi(h(t))\varphi(t)$ is equilateral, and the points $\pm\varphi(t)$, $\pm\varphi(h(t))$, $\pm\varphi(h(h(t)))$ form a hexagon that is inscribed into C. Let further on $F : \mathbb{R} \to \mathbb{R}$ be a monotone function satisfying the equation $F(h(t)) = F(t) + \frac{\pi}{3}$. Then the requested angular measure is obtained by

$$\sphericalangle (\varphi(t), \varphi(s)) := F(s) - F(t),$$

where $s, t \in \mathbb{R},\ t \leq s \leq t + 2\pi$.

For the case that B contains a segment of length > 1, a projection $B \to B^*$ is introduced such that

- B^* is the unit disc of another plane,
- equilateral triangles with respect to B are mapped onto equilateral triangles with respect to B^*, and
- B^* contains less segments of length > 1 than B.

Finally it is shown that each unit ball contains at most two pairs of segments of length > 1, which completes the proof. □

Our goal is to achieve similar results. We declare any two angles, that fulfill a certain property, to be equal and derive characterisations of planes where such an angular measure exists. Ideas for getting appropriate properties can be taken from the large amount of Euclidean results dealing with equal angles or angles of certain value. For example, we have

- the fact, that base angles in isosceles triangles are of equal size,

- the inscribed angles theorem, in particular Thales' theorem, and
- the fact, that angles between orthogonal lines are of equal value, namely $\frac{\pi}{2}$.

2.4 I-measures

In this section we will deal with so called *I-measures*, where the "I" stands for "isosceles". The definition will clarify the choice of this denotation. But first, we want to state the following

Proposition 2 *Let μ be an angular measure in a Minkowski plane. The following conditions are equivalent:*

$i)$ For any isosceles triangle $\triangle abc$ the corresponding base angles are equal, i.e.,

$$||a-c|| = ||b-c|| \Rightarrow \sphericalangle(c,a,b) = \sphericalangle(c,b,a).$$

$ii)$ Let $a,b,c \in S$ be pairwise distinct. If $c \notin \widehat{ab}$, then

$$\sphericalangle(a,b) = 2 \cdot \sphericalangle(a,c,b);$$

otherwise

$$2\pi - \sphericalangle(a,b) = 2 \cdot \sphericalangle(a,c,b).$$

$iii)$ Let $a,b \in S$ be distinct and $c := -a$. Then

$$\sphericalangle(a,b) = 2 \cdot \sphericalangle(a,c,b).$$

Proof: $\underline{i) \Rightarrow ii)}$: This direction can be proved in the same way as in the Euclidean case; see [11], p. 46.

$\underline{ii) \Rightarrow iii)}$: This conclusion is obviously true.

$\underline{iii) \Rightarrow i)}$: Due to basic properties of angles we have

$$\sphericalangle(a,b) = \pi - \sphericalangle(b,c) = \sphericalangle(o,b,c) + \sphericalangle(o,c,b).$$

Since $\sphericalangle(a,b) = 2\cdot\sphericalangle(a,c,b)$, it follows that $\sphericalangle(o,c,b) = \sphericalangle(o,b,c)$ for any $a,b \in S$. □

<u>Remark:</u> In the Euclidean plane, property *ii*) of Proposition 2 is known as the *inscribed angle theorem*, which is very often used in proofs of many theorems (for example, for proving the Theorem of Thales and, as a corollary, the fact that cyclic quadrilaterals are exactly the quadrilaterals where opposite angles sum up to π).

Definition 4 *Let μ be an angular measure in a Minkowski plane. If μ satisfies one of the conditions in Proposition 2, then μ is said to be an* I-measure.

The denotation is derived from the property that the measure yields a special property for angles in isosceles triangles, namely that the base angles are of equal value.

In what follows, $M^2(B)$ is always a plane where an I-measure μ exists, and angles are always measured by μ.

Proposition 3 *There exist no non-trivial zero angles, i.e.,*

$$\sphericalangle(u,v) > 0 \quad \forall u,v \in S, u \neq v.$$

Proof: Suppose $\sphericalangle(u,v) = 0$ for some $u,v \in S,\ u \neq v$. Without loss of generality, $\sphericalangle(u',v') > 0$ for all angles $\sphericalangle(u',v')$ strictly containing $\sphericalangle(u,v)$. This holds since, if $\sphericalangle(u_n,v_n)$ is a sequence of angles of measure 0 that is increasing with respect to $\subseteq$, and $u_n \to \overline{u}$, $v_n \to \overline{v}$, then $\sphericalangle(\overline{u},\overline{v}) = \sup \sphericalangle(u,v) = 0$.

Now we translate the angle $\sphericalangle(u,v)$ so that its apex moves into S and such that the translated angle $\sphericalangle(u',o',v')$ contains $\sphericalangle(u,v)$; see Figure 7.

Then $\sphericalangle(u',v') = 2\cdot\angle(u,v) = 0$, but $\sphericalangle(u',v')$ strictly contains $\sphericalangle(u,v)$, a contradiction. □

The following statement is a direct consequence of Proposition 2.

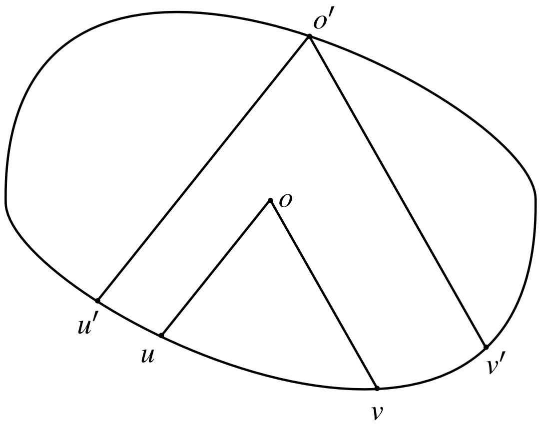

Figure 7 – Notation in the proof of Proposition 3.

Lemma 2 *Thales' theorem holds, i.e., for all* $u, v \in S,\ u \neq v$, *we have*

$$\sphericalangle\,(-uvu) = \frac{\pi}{2}.$$

Proof: Let μ be an I-measure in $M^2(B)$. Thus, property *ii)* of Proposition 2 holds. Let $a = u$, $b = -u$ and $c = v$. Then $2\sphericalangle\,(-u, v, u) = \sphericalangle\,(-u, u) = 2\pi$. □

Lemma 3 *Let* $p \in S$, *and let* t *be a supporting line of* B *at* p. *Then* $\sphericalangle\,(o, p, q) = \frac{\pi}{2}$ *for* $q \in t \setminus \{p\}$. *In other words: If any two vectors are Birkhoff orthogonal, then they enclose an angle of measure* $\frac{\pi}{2}$.

Proof: The statement follows from the previous lemma, taking into consideration the continuity of the measure. Let $p \in S$ be an arbitrary point on the unit circle, and let $\{s_k\} \subset S,\ k = 0, 1, \ldots$, be a sequence of points with $\lim_{k\to\infty} s_k = p$. Then $\sphericalangle\,(o, p, s_k) \to \sphericalangle\,(o, p, q)$ and $\sphericalangle\,(p, o, s_k) \to 0$. Since $\sphericalangle\,(-p, s_k, p) = \frac{\pi}{2}$ for all k and the formula for the sum of inner angles holds for triangles, we have that $\sphericalangle\,(o, p, s_k) \to \frac{\pi}{2}$, which completes the proof. □

Finally we are able to prove the main theorem of this section. This theorem clarifies in which planes there exists an I-measure.

Theorem 2 *Let $M^2(B)$ be a Minkowski plane with unit circle S, and μ be an I-measure on S. Then $M^2(B)$ is the Euclidean plane, and μ is the Euclidean angular measure.*

Proof: Let $x \in M^2(B)$ be arbitrary, and $y \in M^2(B)$ be such that $x \perp_I y$. Then Lemma 1 and Lemma 2 yield $\sphericalangle(x, y) = \frac{\pi}{2}$.

Now let $z \in M^2(B)$ be such that $||z|| = ||y||$ and $x \perp_B z$. Then, by Lemma 3, we have that $\sphericalangle(x, z) = \frac{\pi}{2}$. Since there exist no non-trivial zero angles, it follows that $y = z$ or $y = -z$, and thus we have

$$x \perp_I y \Rightarrow x \perp_B y \quad \forall x, y \in M^2(B).$$

As we have already stated, this relation is a characterisation of the Euclidean plane; see section 1.2.3.

It remains to show that the measure is the Euclidean one. By definition, a straight angle has value π. Proposition 2 on page 17 yields that the measure of a 2^{-k}-multiple of a straight angle is equal to the Euclidean value of the respective angle for all $k \in \mathbb{N}$. Let $x, y \in S$ arbitrary, and let a be the Euclidean value of $\sphericalangle(x, y)$. Since a can be decomposed uniquely into a sum of powers of $1/2$, it follows with the additivity of the measure μ that $\sphericalangle(x, y) = a$. □

2.5 B-measures

The "B" in *B-measure* refers to Birkhoff orthogonality, and the following definition clarifies the choice of this denotation.

Definition 5 *An angular measure μ is said to be a B-*measure *if the value of the angle between two Birkhoff orthogonal vectors is $\frac{\pi}{2}$, i.e., if*

$$x \perp_B y \Rightarrow \sphericalangle(x, y) = \frac{\pi}{2} \quad \forall x, y \in M^2(B). \tag{1}$$

We can immediately state the following

Observation: Let $x, y, z \in S$ be such that $x \perp_B y$ and $y \perp_B z$ (and thus $y \perp_B -z$). Then one of the angles $\sphericalangle(x,z)$ and $\sphericalangle(x,-z)$ is a zero angle, and the other one a straight angle.

Proof: By additivity we have $\sphericalangle(x,z) = \sphericalangle(x,y) + \sphericalangle(y,\pm z) = \frac{\pi}{2} \pm \frac{\pi}{2}$. □

From now on we will need a parametrisation of the unit circle. Consider therefore a function $\varphi : \mathbb{R} \to S$ with the property that

$$\varphi(t) = \varphi(t + 2\pi) \quad \forall t \in \mathbb{R}.$$

In this way, every $t \in \mathbb{R}$ is uniquely associated with a point $x = \varphi(t) \in S$. If it is clear from the context what is meant, we sometimes simply write t instead of $\varphi(t)$. We say that $s, t \in \mathbb{R}$ are orthogonal if $\varphi(s)$ is orthogonal to $\varphi(t)$.

Additionally, we will need the notion of *Radon points*, which directly yields further important notions; see the following

Definition 6 *Let $M^2(B)$ be a Minkowski plane. An interval $R \subset \mathbb{R}$ is called a* Radon interval *if for all $s \in R$, $t \in \mathbb{R}$, with $s \perp_B t$ we have that $t \perp_B s$. The corresponding arc $\varphi(R)$ is called a* Radon arc.

We call a point $s \in \mathbb{R}$ or $x \in S$ a Radon point *if it is contained in a Radon interval or a Radon arc, respectively. If every point $x \in S$ is a Radon point, then S is called a* Radon curve *and $M^2(B)$ is a* Radon plane.

Remark: Let R be a Radon arc of S. Then $-R$ and $R' := \{y \in S : \exists x \in R$ such that $x \perp_B y\}$ are also Radon arcs of S.

A point that is not a Radon point is called a *non-Radon point*. An interval (arc) absolutely free of Radon points is said to be a *non-Radon interval (arc)*, and a plane that is not a Radon plane is called a *non-Radon plane*.

Lemma 4 *Let $M^2(B)$ be a non-Radon plane, but assume that its unit circle contains a Radon arc, and let $u, v \in \mathbb{R}$ with $u < v < u + \pi$ be such that $N := [u, v]$ is a non-Radon interval. Then $\sphericalangle(\varphi(u), \varphi(v)) = 0$.*

Proof: Let $x_0 \in N$, $x_0 \perp_B p_0$, and $x_1 \in [x_0 - \pi, x_0 + \pi]$ be such that $p_0 \perp_B x_1$, and that the Euclidean measure of $\sphericalangle(\varphi(x_0), \varphi(x_1))$ is maximal (for non-smooth Minkowski planes x_1 need not be unique). Without loss of generality, $x_1 > x_0$. Then $[x_0, x_1]$ is a non-Radon interval.

Contrarily, assume that there exist $y \in [x_0, x_1]$ and $z \in [y, y + \pi]$ with $y \perp_B z$ and $z \perp_B y$ (and thus y is a Radon point). Since B is convex, the slopes of the supporting lines at B are monotone. Thus we have

$$y > x_0 \Rightarrow z > p_0, \text{ as } x_0 \perp_B p_0, \text{ and}$$
$$z > p_0 \Rightarrow y > x_1, \text{ as } p_0 \perp_B x_1,$$

a contradiction.

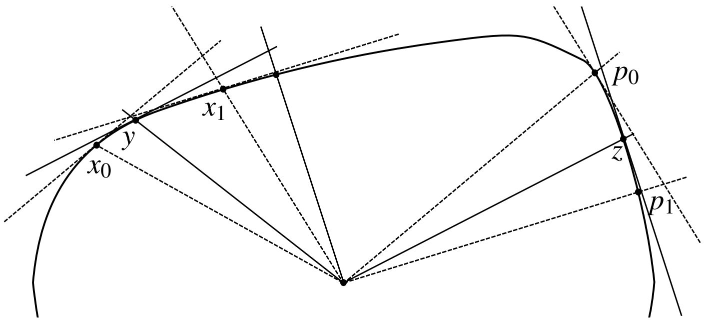

Figure 8 – Zero angles in Non-Radon planes.

In this way we can construct a sequence $\{x_k\}_{k\geq 0}$ with the following properties:

$$x_k \in N,\ x_{k+1} > x_k, \text{ and } \sphericalangle(\varphi(x_k), \varphi(x_{k+1})) = 0 \quad \forall k \geq 0.$$

Analogously, we define a sequence $\{x_{-k}\}_{k\geq 0}$ which we get by applying Birkhoff orthogonality in the opposite direction. We have that $\sphericalangle(\varphi(x_{-k}), \varphi(x_{-k-1})) = 0$ $\forall k \geq 0$. Thus both these sequences have to converge. Let

$$x^+ := \lim_{k\to\infty} x_k, \quad x^- := \lim_{k\to-\infty} x_k.$$

Since $\{x_k\}_{k\geq 0}$ is strictly increasing, we have that $x^+ > x_k$ for all $k \geq 0$, and thus x^+ is a Radon point. (Otherwise the construction above would yield a point x^{++} with

$x^{++} \in N$ and $x^{++} > x^{+}$, a contradiction to the fact that x^{+} is the limit of the sequence $\{x_k\}_{k\geq 0}$.) Analogously, we conclude that x^{-} is a Radon point and that $N \subseteq [x^{+}, x^{-}]$. By continuity of μ we get $\sphericalangle(\varphi(x^{+}), \varphi(x^{-})) = 0$. □

Remark: Figure 9 gives an example for a non-Radon plane whose unit circle contains Radon arcs. This example clarifies that the formulation in Lemma 4 makes sense.

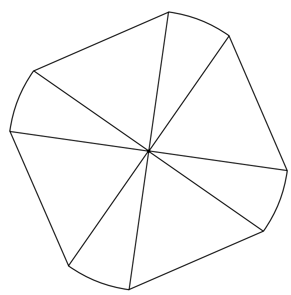

Figure 9 – Example for the unit disc B of a non-Radon plane containing Radon-arcs.

Except for the mid- and endpoints, the straight line segments of the boundary of B consist of non-Radon points. On the other hand, the curve segments are parts of a Euclidean circle and thus consist of Radon points.

Theorem 3 *Let $M^2(B)$ be a Minkowski plane with unit circle S. There exists a B-measure on S if S contains a Radon arc.*

Proof: To show that there exists a B-measure on S, we construct one, and we do this in a way similar to that of Brass [7]; see section 2.3 on page 14. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous, monotonously increasing function with the properties

$$t < h(t) < t + \pi \text{ and } t \perp_B h(t) \quad \forall t \in \mathbb{R}.$$

We consider an arbitrary monotone function $F : \mathbb{R} \to \mathbb{R}$ such that the equation $F(h(t)) = F(t) + \frac{\pi}{2}$ is satisfied, and that if S contains a non-Radon arc with corresponding non-Radon interval N the equation $F(s) - F(t) = 0$ holds for $s, t \in N$, $0 \leq s - t < \pi$. Then the angular measure satisfying (1) is obtained by

$$\sphericalangle(\varphi(t), \varphi(s)) := F(s) - F(t),$$

where $s, t \in \mathbb{R},\ t < s \leq t + 2\pi$. □

As an immediate consequence of this theorem we get a characterisation of Radon planes.

Corollary 1 *Let $M^2(B)$ be a Minkowski plane with unit circle S such that S contains a Radon arc. Then $M^2(B)$ is a Radon plane if and only if there exists a B-measure on S that is strictly increasing.*

Proof: If S contains a non-Radon arc, the construction above shows that the function F has to be constant on every non-Radon interval. If S is a Radon plane, we can choose F as strictly increasing. □

2.6 T-measures

Definition 7 *An angular measure μ is called* T-measure *if the value of the angle between two James-orthogonal vectors is $\frac{\pi}{2}$, i.e., if*

$$x \perp_J y \Rightarrow \sphericalangle(x, y) = \frac{\pi}{2} \quad \forall x, y \in M^2(B). \tag{2}$$

As shown in Lemma 1 on page 14, this condition is equivalent to the fact that the theorem of Thales holds, i.e.,

$$\sphericalangle(-u, v, u) = \frac{\pi}{2} \quad \forall u, v \in S,\ u \neq \pm v.$$

Remark: We have that every T-measure is a B-measure. Namely, Lemma 3 on page 19 states that for every measure that fulfils the previous condition, property (1) is fulfilled. In this sense, condition (1) is more general than (2). As a consequence, we expect that T-measures are "less common" than B-measures in the sense that they do not exist in every plane that contains a Radon arc. The following theorem confirms this conjecture.

Theorem 4 *Let $M^2(B)$ be a Minkowski plane with unit circle S. A T-measure on S exists if and only if $M^2(B)$ is the Euclidean plane.*

To prove this theorem, we first need some measure theory.

Definition 8 *For a point $x \in \mathbb{R}$ and $\varepsilon > 0$ we define*

- *a* lefthand interval *as an interval of the form $[x - \varepsilon, x]$,*
- *a* righthand interval *as an interval of the form $[x, x + \varepsilon]$.*

Lemma 5 *Let η be a continuous measure on $[0, 2\pi)$ with $\eta([0, 2\pi)) = 2\pi$. (The interval $[0, 2\pi)$ can be considered as the preimage of S.) Let G be the set of points for which all lefthand intervals, as well as all righthand intervals, have positive η-measure. Then $\eta([0, 2\pi)\backslash G) = 0$.*

Remark: We call a point $x \in \mathbb{R}$ a *good point (with respect to η)* if $x - k \cdot 2\pi \in [0, 2\pi)$ fulfils the condition of Lemma 5; otherwise x is said to be a *bad point (with respect to η)*. We want to show that there are only countably many bad points. In view of this it is not essential that a priori multiples of 2π have no lefthand intervals.

Proof: The *support* of η is defined by

$$\operatorname{supp}(\eta) := \{x \in [0, 2\pi) \mid \eta(N_x) > 0 \text{ for any neighbourhood } N_x \text{ of } x\}.$$

Then $\operatorname{supp}^C(\eta)$, the complement of the support $\operatorname{supp}(\eta)$, consists of at most countably many open sets:

Since $\text{supp}(\eta)$ is closed, $\text{supp}^C(\eta)$ is open. Let $x \in \text{supp}^C(\eta)$. Then there exists an interval $I \subseteq [0, 2\pi)$ such that $x \in I \subseteq \text{supp}^C(\eta)$. We consider an interval of maximal length with this property. Different points of $\text{supp}^C(\eta)$ yield either the same interval or disjoint intervals. Since each of these intervals contains a rational point, there exist only countably many of them.

Let now $x \in [0, 2\pi)$ be a bad point. Then either some righthand intervals or some lefthand intervals have measure 0, i.e., x is boundary point of one of the maximal open intervals in $\text{supp}^C(\eta)$. Thus we have that there exist at most countably many bad points. Since η is continuous, these points have measure zero. □

Now we are able to prove the theorem.

Proof of Theorem 4: Let $x \in S$ be the image of a good point with respect to μ under φ, and let $w \in S$ be such that $x \perp_B w$. Then $\sphericalangle(w, x) = \frac{\pi}{2}$.

Furthermore, let $u, v \in S$ be such that $v - u$ is a multiple of w, i.e., $(v - u) \parallel w$. We consider the line passing through o and $\frac{u+v}{2}$. For $k = \left\|\frac{v-u}{2}\right\|$ we get

$$u = \frac{v+u}{2} - k \cdot \frac{w}{\|w\|} \text{ and } v = \frac{v+u}{2} + k \cdot \frac{w}{\|w\|}.$$

Since $\|u\| = \|v\| = 1$, we have $\left(\frac{v+u}{2}\right) \perp_I \left(\frac{k \cdot w}{\|w\|}\right)$, and thus $\sphericalangle\left(\frac{v+u}{2}, w\right) = \frac{\pi}{2}$.

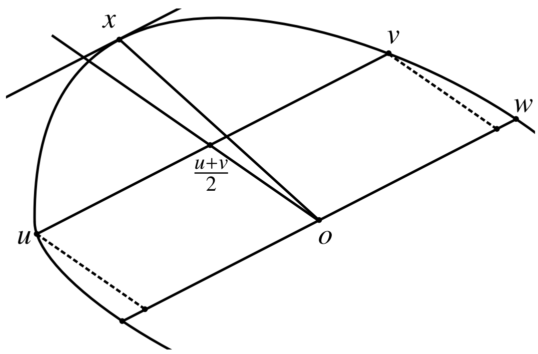

Figure 10 – Notation in the proof of Theorem 4.

Since x is a good point, there are no zero angles in a neighbourhood of x, and therefore all the midpoints of chords of S parallel to w lie on the line passing through o

and x. We call this line an *(affine) symmetry axis* of S, and we call u the *reflection point of v with respect to* (o, x).

We show that every line through the origin is a symmetry axis of S. Then by [40, Th. 3.4.1] S is an ellipse with centre o, and thus $M^2(B)$ is Euclidean.

Let $x \in S$ be a good point. Then by definition (o, x) is a symmetry axis of S. Let, on the other hand, $x \in S$ be a bad point. Then for every good point $z \in S \backslash \{-x\}$ there exists a reflection point y_z of x with respect to (o, z). As there are at most countably many bad points, there exists a good point g such that y_g is a good point, too. Hence $(0, x)$ is the reflection image of the symmetry axis $(0, y_g)$ (with respect to (o, g)), and thus $(0, x)$ is itself a symmetry axis.

It remains to show, that there exists a T-measure on S if $M^2(B)$ is the Euclidean plane. But obviously the Euclidean measure fulfils condition (2). □

Remark: Although the plane has to be Euclidean, the angular measure need not necessarily be the Euclidean one. Strictly speaking, any B-measure referring to the Euclidean circle fulfils property (2).

2.7 Angle bisectors

It is natural to ask whether there are immediate possibilities to define angle bisectors. Obviously, for a given measure μ one can instantly specify the bisector of each angle, the so-called *μ-bisector*. The question becomes more interesting if we demand that the bisectors fulfill special properties.

For instance, N. Düvelmeyer [10] gave some nice results for so-called Busemann and Glogovskij angular bisectors. Given $u, v \in S$, the *Busemann angular bisector* of the angle $\sphericalangle(u, v)$ is defined by $A_B(\sphericalangle(u, v)) = \{k(u + v),\ k \geq 0\}$, and the *Glogovskij angular bisector* is defined as the set of points that have equal distance from the lines (o, u) and (o, v). Düvelmeyer then proved that these two bisector definitions coincide if and only if the plane under consideration is a Radon plane. Further on, if any of them coincides with the μ-bisector of a given measure μ, then the plane has to be Euclidean.

We will have a look at μ-bisectors with respect to a Brass-measure, i.e., a measure where the angles of an equilateral triangle are all of equal value. We obtain the following interesting result.

Theorem 5 *Let $M^2(B)$ be any Minkowski plane except for the rectangular one; i.e., a plane where an angular measure in the sense of Brass exists. Then there exists an angular measure μ such that the inner angles of every equilateral triangle have a value of $\frac{\pi}{3}$, and in every equilateral triangle the bisectors of the inner angles intersect in a common point.*

Proof: Let T be an equilateral triangle, and H be an equilateral hexagon inscribed to B such that the sides of H are parallel to the sides of T. Our goal is to find a second inscribed equilateral hexagon H^* with the property that the sides of H^*, attached to the vertices of T, intersect in one point. Let x be a vertex of H, and let at first $x' = x$. Then we move x' in positive orientation and consider the inscribed hexagon H' that has x' as a vertex. For every x', we take lines parallel to the sides of H' and attach them to the vertices of T; see Figure 11. If for a certain point x' there is no unique inscribed hexagon, we fix x' and $-x'$ for the moment and move only the other four vertices (in the same direction as before); see Figure 6 on page 16.

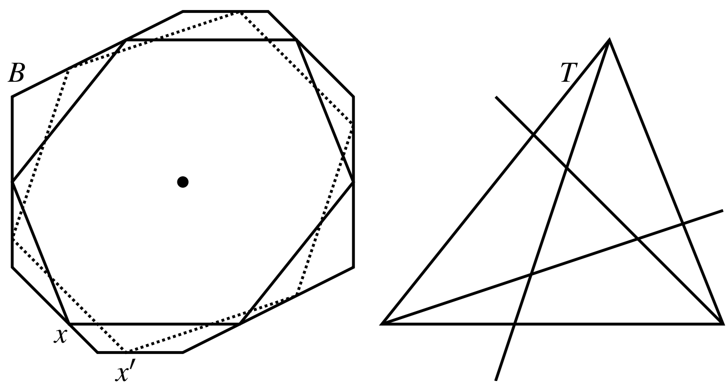

Figure 11 – The “pointed” hexagon does not fulfill the required property; x' still has to move a bit.

When x' (and thus H') moves continuously, the lines attached to the vertices of T move continuously between the two sides of T containing these vertices. Thus there exists a set of directions (i.e., an appropriate hexagon H^*) such that all three lines intersect in one point. □

Remark: This result does not give a construction of the appropriate hexagon H^*, it only proves its existence. Moreover, it does only fix angles of value $\frac{\pi}{6}$, not general angles. In other words, the theorem says that there exists a possibility to partition the unit circle into 12 equal parts with the property mentioned above.

2.8 Outlook

There are various possibilities to extend the concepts discussed above. On the one hand, one may apply the idea, that "the angle between two orthogonal vectors equals $\frac{\pi}{2}$" to other orthogonality types, e.g., the Singer orthogonality, Pythagorean orthogonality et al.; see [1, 2] for detailed information.

On the other hand, one can study how a plane or an angular measure have to look like if the straight angle is divided into $n \in \mathbb{N}$ equal parts. The case where $n = 2$ yields the B-measure, and $n = 3$ is the topic investigated by Brass [7]. For $n = 4$ we would get something like angle bisectors for the B-measure, and for $n = 6$ the same thing for the measure that was considered by Brass; see the last section.

3 Types of convex quadrilaterals

3.1 Introduction and Definition

It is well known that already notions like regular, equilateral or isosceles triangles in normed planes yield interesting geometric problems and applications; see, e.g., [28], §5. The same holds for special types of convex quadrilaterals in normed planes (cf. [28], §§3.3, and [30]), but no complete classification of possible types of convex quadrilaterals seems to be known. In this chapter we define various classes of such 4-gons and investigate relations between them. The main types of quadrilaterals under consideration are (Minkowskian) rectangles, rhombi, and so-called codises, but we continue in giving even a finer structure. The results of this chapter are submitted [16].

Definition 9 *Let $M^2(B)$ be a Minkowski plane. A subset Q of $M^2(B)$ is called a* (convex, non-degenerate) quadrilateral *if it is the convex hull of four points in convex position, i.e., none of these four points is contained in the convex hull of the other three. We call these four points the* vertices *of Q. The four segments bounding Q are called its* sides*, and the two segments that join two of the four points, but are not sides of Q, are called the* diagonals *of Q. The quadrilateral Q is said to have* halving diagonals *if the common point of the two diagonals is the midpoint of each of them.*

Remark: Sometimes when we write “quadrilateral” we refer only to the boundary of Q. It should always be clear from the context what we mean.

3.2 Parallelograms

We use the notion introduced in [28].

Definition 10 *We call a quadrilateral Q in $M^2(B)$ a* metric parallelogram *if each two opposite sides are of equal length.*

Theorem 6 *Let Q be a quadrilateral in $M^2(B)$, and consider the following properties:*

i) Opposite sides of Q are parallel.

ii) Q has halving diagonals.

iii) Q is a metric parallelogram.

Then i) $\Leftrightarrow$ ii) $\Rightarrow$ iii), and iii) $\Rightarrow$ i) if and only if the plane is strictly convex.

Proof: Let Q be a quadrilateral with vertices o, a and b. Opposite sides of Q are parallel if the fourth vertex is, e.g., $a+b$. Then the midpoint of $[0, a+b]$ is $\frac{1}{2}(a+b)$, and the midpoint of $[a,b]$ is $a + \frac{1}{2}(b-a) = \frac{1}{2}(a+b)$. Hence the midpoints of the diagonals of Q coincide.

Let, on the other hand, the vertices of Q be a, b, $-a$, and $-b$ (thus, the diagonals halve each other and intersect in o). Then, obviously, $[a,b]$ is a translate of $[-b,-a]$, and $[b,-a]$ is a translate of $[a,-b]$. Thus, they are of equal length and parallel.

The second statement can be found in [41]. It uses mainly the fact, that two Minkowskian circles in strictly convex planes intersect in at most two points. Figure 12 shows a counterexample for a plane that is not strictly convex. $\square$

3.3 Rectangles

In the Eulidean plane, a rectangle is a quadrilateral that has halving diagonals which are of equal length or, analogously, has the property that neighbourly sides are

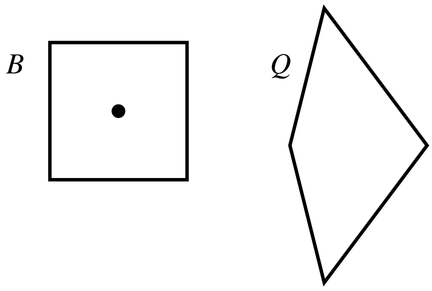

Figure 12 – Q is a metric parallelogram, but has no parallel sides.

orthogonal. In arbitrary Minkowski planes we have to distinguish between these definitions.

Definition 11 *We call a quadrilateral Q in $M^2(B)$ a* rectangle defined by its diagonals*, abbreviated by* d-rectangle*, if its diagonals halve each other and are of equal length, and we call it a* rectangle defined by isosceles-orthogonal sides *or* i-rectangle*, if each two adjacent sides are isosceles orthogonal.*

We have immediately the following

Observation: Every d-rectangle is an i-rectangle.

It is easy to see that, in general, the opposite is not true; see Figure 13 for an example. Though the diagonals of Q are of equal length, they do not halve each other.

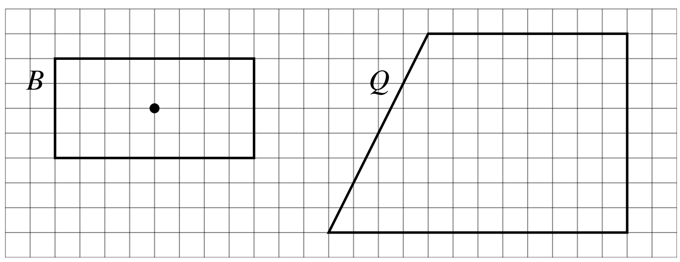

Figure 13 – Q is an i-rectangle, but not a d-rectangle.

We want to discuss under what conditions every i-rectangle is a d-rectangle. Obviously, the following is true.

Proposition 4 *In a strictly convex Minkowski plane, every i-rectangle, which has two pairs of opposite sides of equal length, is a d-rectangle.*

Proof: By Theorem 6 every i-rectangle with that property is a parallelogram, which means that its diagonals halve each other. Moreover, due to i-orthogonality of the sides, they are of equal length. $\square$

In fact, each of the two conditions suffices to achieve the same result.

Theorem 7 *i) If $M^2(B)$ is a strictly convex Minkowski plane, then every i-rectangle is a d-rectangle.*

ii) Every i-rectangle, which has two pairs of opposite sides of equal length, is a d-rectangle.

Proof: i) Let $M^2(B)$ be a strictly convex Minkowski plane, and let Q be an i-rectangle with vertices a, b, c, and d; without loss of generality, let $c = -a$. Since Q is an i-rectangle, $b - a \perp_I c - b$, and hence $b - a \perp_I a + b$, which yields $||a|| = ||b||$. Analogously, from $d - a \perp_I c - d$ it follows that $||a|| = ||d||$. Since $||a|| = ||c||$, o is a circumcenter of Q, i.e., the midpoint of a circle that contains all four vertices in its boundary.

With the same arguments we get that the midpoint $\frac{1}{2}(b + d)$ of the second diagonal is a circumcenter of Q, too. Since the plane is strictly convex, every triangle has exactly one circumcenter, and hence every quadrilateral has at most one; see [40], p. 104. Thus the diagonals of Q halve each other.

ii) Let Q be an i-rectangle with sides a, b, c, and d, where $||a|| = ||c||$ and $||b|| = ||d||$. Then Remark 2.2 in [25] yields that $a \parallel c$ and $b \parallel d$. Thus Q is a parallelogram, which means that its diagonals are $a + b$ and $a - b$. Due to i-orthogonality of the sides the diagonals are then of equal length. By Theorem 6 they halve each other. $\square$

3.4 Rhombi

Analogously to rectangles we have to distinguish between quadrilaterals with orthogonal and halving diagonals and quadrilaterals with sides of equal length, which is equivalent in the Euclidean plane.

Definition 12 *We call a quadrilateral Q in $M^2(B)$ a* rhombus defined by its sides*, abbreviated by* s-rhombus*, if its sides are of equal length, and we call it a* rhombus defined by isosceles-orthogonal diagonals *or* i-rhombus*, if its diagonals halve each other and are isosceles orthogonal.*

Observation: Every i-rhombus is an s-rhombus.

Again, we ask whether the opposite is true. Figure 12 shows that it is in general not true. However, we have the following

Theorem 8 *In a strictly convex Minkowski plane, every s-rhombus is an i-rhombus.*

Proof: By definition, s-rhombi have two pairs of opposite sides of equal length. Since we are in a strictly convex plane, by Theorem 6 every s-rhombus is a parallelogram. Thus its diagonals are i-orthogonal (by definition). □

3.5 Sets of constant distance sum

Besides rectangles and rhombi, another type of convex quadrilaterals can be introduced.

Definition 13 *Let $c > 0$ and let l_1 and l_2 be two lines, where $l_1 \nparallel l_2$; let these lines, without loss of generality, intersect in o. The set of points, whose sum of distances to the lines l_1 and l_2 equals c, is called a* set of constant distance sum*, abbreviated* codis*, with respect to l_1, l_2.*

It is not clear from the definition whether every set of this kind is the convex hull of four points. But in fact, we have the following

Lemma 6 *Every codis is a convex quadrilateral.*

Proof: Let $a \in l_1$ and $b \in l_2$ be two points such that $d(a,\ l_2) = d(b,\ l_1) = c$. (Thus, a and b belong to the same codis.) We show that every point of the line segment $[a, b]$ belongs to this codis, too; i.e., that

$$d(x_\lambda,\ l_1) + d(x_\lambda,\ l_2) = c \quad \forall \lambda \in [0, 1],$$

where $x_\lambda = a + \lambda \cdot (b - a)$; see Figure 14.

Let l_1' be a line parallel to l_1 and passing through x_λ. Let y be the intersection point of l_1' and l_2. Then the triangle $\Delta x_\lambda y b$ is similar to the triangle Δaob. Since $d(a,\ l_2) = d(b,\ l_1)$, we have that $d(x_\lambda,\ l_2) = d(b,\ l_1')$. Additionally, $d(x_\lambda,\ l_1) = d(y,\ l_1)$. Thus

$$d(x_\lambda,\ l_1) + d(x_\lambda,\ l_2) = d(y,\ l_1) + d(b,\ l_1') = d(b, l_2) = c.$$

□

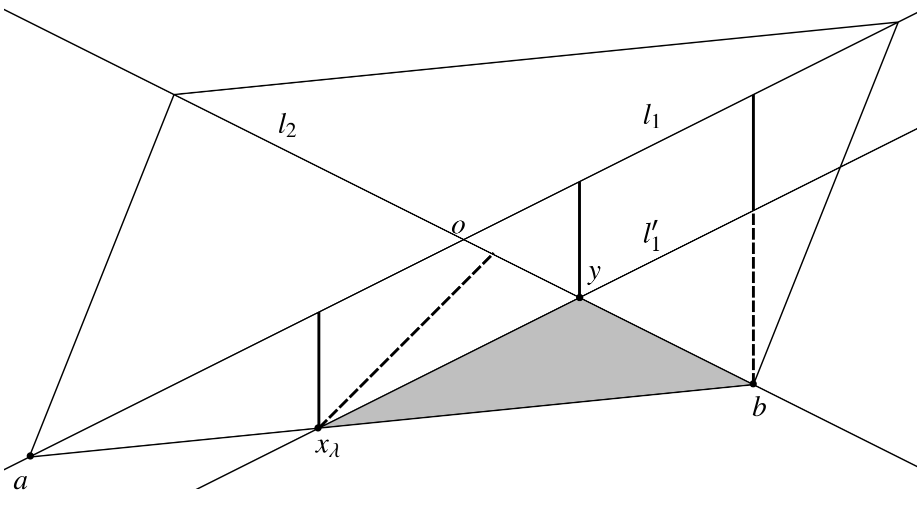

Figure 14 – Every codis is a quadrilateral.

From the proof, we get immediately the following

Corollary 2 *Every codis is a parallelogram.*

Our next theorem shows that in the Euclidean plane, as well as in any other Radon plane, codises do not exist as an "own type of quadrangle".

Theorem 9 *Let $M^2(B)$ be a Radon plane. Then every codis in $M^2(B)$ is a d-rectangle, and every d-rectangle is a codis.*

Proof: Let Q be a parallelogram with vertices a, b, c, and d. Let the length of the side $[a, b]$ be g, let the distance from c and d to the line (a, b) be h, let the lengths of the diagonals be $e = ||a - c||$ and $f = ||b - d||$, and let the distances from a to (b, d) and from b to (a, c) be d_1 and d_2, respectively; see Figure 15.

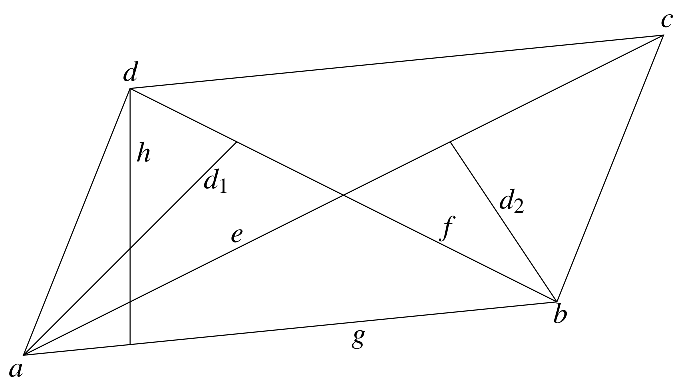

Figure 15 – Notation in the proof of Theorem 9.

By [28], p. 124, we have that a Minkowski plane is a Radon plane if and only if the following property holds: For every triangle with side lengths a, b, and c and respective heights h_a, h_b, and h_c, where the height is the value of a shortest distance from a vertex to the opposite side, the following equality holds:

$$a \cdot h_a = b \cdot h_b = c \cdot h_c.$$

Applying this equality to the triangle Δabd we get $d_1 \cdot f = h \cdot g$. Analogously, in the triangle Δabc we have $d_2 \cdot e = h \cdot g$. Thus we have $d_1 = d_2 \iff e = f$. $\square$

We conjecture that the opposite is also true. At least we can give an example of a plane (that, of course, is not a Radon plane) and a quadrilateral Q that is a d-rectangle (even a square, in the sense of four sides of equal length), but not a codis; see Figure 16. Obviously, Q is equilateral, its diagonals are of equal length and halve each other. But the distance from the lower left corner to the diagonal not containing it is 1, whereas the distance fom the lower right corner to the other diagonal is clearly smaller than 1.

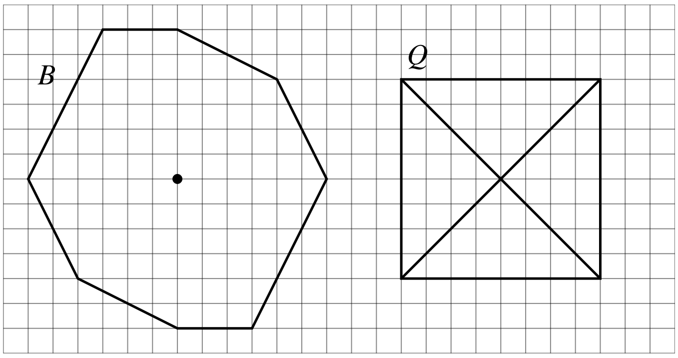

Figure 16 – Q is a d-rectangle, but not a codis.

3.6 Quadrilaterals in Radon planes

We introduce two additional types of quadrilaterals.

Definition 14 *Let $M^2(B)$ be a Radon plane. We call a quadrilateral Q in $M^2(B)$ a* rectangle defined by Birkhoff orthogonal sides *or* B-rectangle *if neighbourly sides of it are pairwise Birkhoff orthogonal, and we call it a* rhombus defined by Birkhoff orthogonal diagonals *or* B-rhombus *if its diagonals halve each other and are Birkhoff orthogonal.*

Remark: We consider only Radon planes, since we want that the orthogonality relation between two sides/diagonals is symmetric.

At first we give some statements that are easy to prove.

Proposition 5 *A Minkowski plane $M^2(B)$ is Euclidean if and only if*

i) every d-rectangle is a B-rectangle,
ii) every s-rhombus is a B-rhombus,
iii) every B-rectangle is a d-rectangle,
iv) every B-rhombus is a s-rhombus.

Proof: $i)$ Let $x,\ y \in S$. Then the quadrilateral with vertices $x,\ y,\ -x$, and $-y$ is a d-rectangle. By assumption, $x+y \perp_B x-y$. Due to Amir [3], (4.2), the plane is then Euclidean.

$ii)$ Let $x,\ y \in S$. Then the quadrilateral with vertices $o,\ x,\ y$, and $x+y$ is an s-rhombus and a parallelogram. Thus its diagonals are $x+y$ and $x-y$, and by assumption $x+y \perp_B x-y$.

$iii)$ Let $x,\ y \in M^2(B)$, where $x \perp_B y$. Then the quadrilateral with vertices $o,\ x,\ y$, and $x+y$ is a B-rectangle and a parallelogramm. Again, its diagonals are $x+y$ and $x-y$, and thus $||x+y|| = ||x-y||$. Then [3], (4.4), completes the proof.

$iv)$ Let $x,\ y \in M^2(B)$, where $x \perp_B y$. Then the quadrilateral with vertices $x,\ y,\ -x$, and $-y$ is a B-rhombus. By assumption, its sides have equal length, i.e., $||x+y|| = ||x-y||$. □

Remark: Due to [3], (10.2), it suffices in $iii)$ (and $iv)$) to consider B-rectangles (B-rhombi) that have sides (diagonals) of equal length.

Not every quadrilateral that is a B-rectangle and an s-rhombus is in general a d-rectangle or a B-rhombus; see Figure 17 for an example.

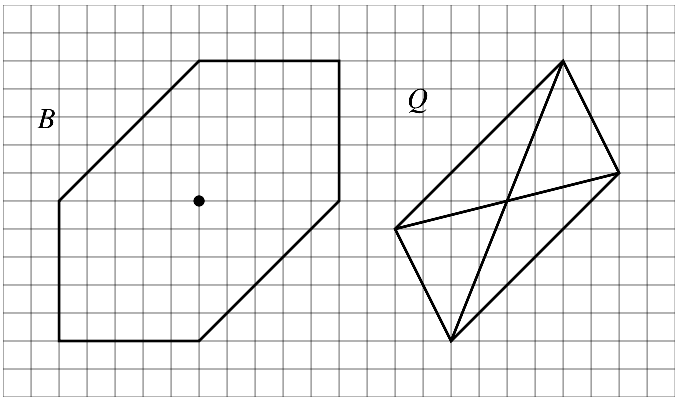

Figure 17 – Q has Birkhoff orthogonal sides of equal length. However, its diagonals are neither Birkhoff orthogonal nor have equal length.

3.7 Associated parallelograms

We introduce the notion of associated parallelograms.

Definition 15 *Let Q be a parallelogram with diagonals e and f. A parallelogram with sides $\frac{\sqrt{2}}{2}e$ and $\frac{\sqrt{2}}{2}f$ is called an* associated parallelogram of *Q, or* associated to *Q and denoted by $\tilde{Q}$.*

Proposition 6 *Generating an associated parallelogram is involutory, i.e., Q is associated to $\tilde{Q}$; see Figure 18.*

Proof: Clearly, the diagonals of $\tilde{Q}$ are parallel to the sides of Q. Thus, the shaded triangles are similar. As the sides of $\tilde{Q}$ have lengths $\frac{\sqrt{2}}{2}e$ and $\frac{\sqrt{2}}{2}f$, the diagonal has length $\sqrt{2}a$; analogously, the other diagonal has length $\sqrt{2}b$. Thus, the sides of $\tilde{\tilde{Q}}$ have lengths a and b, which means that $\tilde{\tilde{Q}} = Q$. □

It follows directly from the definition that (within the class of parallelograms) i-rhombi are associated to i-rectangles, s-rhombi are associated to d-rectangles and

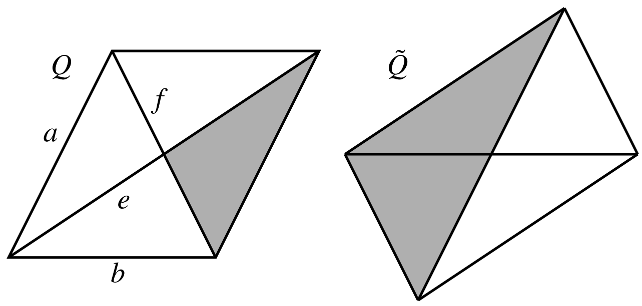

Figure 18 – The associated parallelogram.

B-rhombi are associated to B-rectangles. Thus some (already proved) statements can also be achieved via association:

• *"Every d-rectangle is an i-rectangle."* is associated to *"Every s-rhombus, that is a parallelogram, is an i-rhombus."* On the other hand, *"Every i-rhombus is an s-rhombus."* is associated to *"Every i-rectangle, that is a parallelogram, is a d-rectangle."*

• The statements $i)$ and $ii)$ as well as $iii)$ and $iv)$ of Proposition 5 are associated to each other.

In general, one can say that rectangles are associated to rhombi. We ask whether there is a type of parallelograms associated to codises.

Theorem 10 *If the quadrilateral Q is a codis, then $\tilde{Q}$ has an incircle, i.e., there exists a homothet of C that touches all four sides of $\tilde{Q}$ from inside.*

Proof: Let Q be a codis and let $\tilde{Q}$ be associated to Q such that, without loss of generality, the diagonals of $\tilde{Q}$ intersect in o.

Since Q is a codis, the distances from its vertices to the diagonals not containing them are equal. Thus the distances from o to the sides of $\tilde{Q}$ are equal. In other words, a circle with center o and appropriate radius touches all four sides of $\tilde{Q}$ from inside. $\square$

An immediate consequence of Theorem 9 is the following

Corollary 3 *Let $M^2(B)$ be a Radon plane. Then every i-rhombus in $M^2(B)$ has an incircle, and every parallelogram that has an incircle is an i-rhombus.*

3.8 Squares in Radon planes

Clearly, in the Euclidean definition we call a quadrilateral a *square* if it is a rectangle and a rhombus at the same time.

Definition 16 *Let $\alpha \in \{d, i, B\}$ and $\beta \in \{s, i, B\}$. Then a quadrilateral Q is called an $\alpha\beta$-*square *if it is an α-rectangle and a β-rhombus.*

Lemma 7 *Every square is a parallelogram.*

Proof: It is clear from the definition that all dβ-squares, αi-squares, and αB-squares are parallelograms for all $\alpha \in \{\text{d, i, B}\}$ and $\beta \in \{\text{s, i, B}\}$. By Theorem 7 on page 33, every is-square is a parallelogram. It remains to show that the property holds for Bs-squares.

For this purpose, let $x,\ y,\ z \in S$ be such that $x \perp_B y$ and $x \perp_B z$, where y and z are on the same side of the line (o, x). (Thus, the quadrilateral Q with vertices o, x, y, and $x + z$ is convex; see Figure 19.)

We assume that $y \neq z$. Due to orthogonality, the unit ball is supported by a line parallel to (o, x) in y and in z. Since B is convex, S contains a segment parallel to (o, x) that includes y and z, i.e., $z = y + \alpha \cdot x$ for some $\alpha \in \mathbb{R}$. Thus, the side of Q that is opposite to x is $x + z - y = x + y + \alpha \cdot x - y = x \cdot (1 + \alpha)$. In particular, it is parallel to x. Since Q is equilateral,

$$||x \cdot (1 + \alpha)|| = |1 + \alpha| \overset{!}{=} 1 \Rightarrow \alpha_1 = 0,\ \alpha_2 = -2.$$

The only plane with a unit ball whose boundary contains a segment of length 2 is the rectangular plane, which is not a Radon plane. Thus, $\alpha = 0$, and therefore $y = z$, which implies that Q is a parallelogram. □

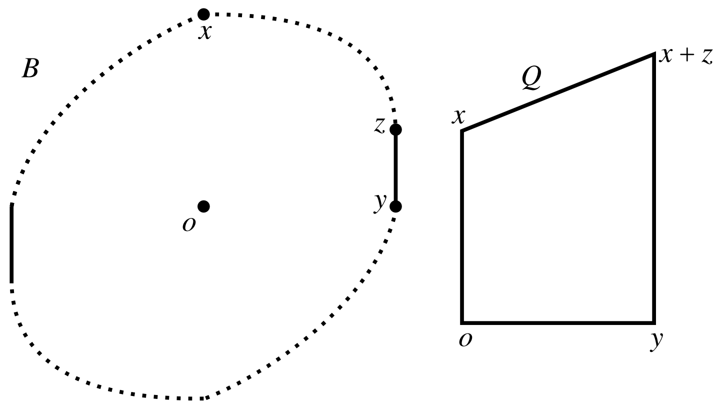

Figure 19 – Notation in the proof of Lemma 7. For the sake of clarity, B and Q are not drawn in the same spot.

Not all combinations of types of rectangles and rhombi yield an own class of squares.

Proposition 7 *Let Q be a quadrilateral.*

$i)$ Q is a ds-square $\iff$ Q is a di-square $\iff$ Q is an ii-square $\iff$ Q is an is-square.
$ii)$ Q is a dB-square $\iff$ Q is an iB-square.
$iii)$ Q is a Bs-square $\iff$ Q is a Bi-square.

Proof: Since every square is a parallelogram, the statements follow from the facts that, within the class of parallelograms, d-rectangles and i-rectangles as well as i-rhombi and s-rhombi coincide (see Theorem 7 and Theorem 8). □

Moreover, these types of squares are distinct unless considered in the Euclidean plane. To prove this, we need the following

Lemma 8 *Let $u, v \in S$ such that $u \neq \pm v$. Then there exists a $k \in \mathbb{R}$ such that $u + k \cdot v \perp_B u - k \cdot v$.*

Proof: For $k = 0$ we have $u + k \cdot v = u - k \cdot v$, for $k \to \infty$ we have $\lim(u + k \cdot v) = -\lim(u - k \cdot v)$. Since $u \pm k \cdot v$ depends continuously on k, there is a k such that $u + k \cdot v \perp_B u - k \cdot v$. □

With this property we are able to prove the announced theorem.

Theorem 11 *If every square of any of the four types (ds-, dB-, Bs-, or BB-square) is also a square of any other type, then the plane $M^2(B)$ is Euclidean.*

Proof: We start by giving a short overview over all four types of squares:

rectangle	rhombus	defined by diagonals	defined by sides
d	s	$\|\|u\|\| = \|\|v\|\|,\ u \perp_I v$	$\|\|u\|\| = \|\|v\|\|,\ u \perp_I v$
d	B	$\|\|u\|\| = \|\|v\|\|,\ u \perp_B v$	$u \perp_I v,\ u + v \perp_B u - v$
B	s	$u \perp_I v,\ u + v \perp_B u - v$	$\|\|u\|\| = \|\|v\|\|,\ u \perp_B v$
B	B	$u \perp_B v,\ u + v \perp_B u - v$	$u \perp_B v,\ u + v \perp_B u - v$

By "defined by diagonals" we mean that the quadrilateral has the vertices $u, v, -u$, and $-v$ (and thus, u and v represent (halves of) the diagonals). Analogously, a quadrilateral that is "defined by sides" has the vertices o, u, v, and $u + v$ (hence, u and v represent sides of the quadrilateral).

We see immediately that dB-squares are associated to Bs-squares. This is why it suffices to consider one of the two types in our proof. BB-squares and ds-squares are self-associated, i.e., if Q is a BB-square (or a ds-square), then $\tilde{Q}$ is of the same type.

i) Let first every ds-square be a dB-square. In other words, for all $u, v \in S$ with $u \perp_I v$ we have that ($u, v \in S$ and) $u \perp_B v$. Thus $M^2(B)$ is Euclidean due to [3], (4.1).

ii) Let every dB-square be a ds-square. Then, analogously, for all $u, v \in S$ with $u \perp_B v$ we have that $u \perp_I v$. This characterises the Euclidean plane due to [3], (4.4).

iii) Let every ds-square be a BB-square; this means that for all $u, v \in S$ with $u \perp_I v$ we have that $u \perp_B v$ and $u + v \perp_B u - v$. As we have seen in *i*), the second property is not necessary to show that the plane is Euclidean.

iv) Let every BB-square be a ds-square. We assume that the plane is not Euclidean and show the existence of two vectors u and v such that $u \perp_B v$, $u+v \perp_B u-v$, but $||u|| \neq ||v||$ or $||u + v|| \neq ||u - v||$ (thus u and v form a BB-square that is not a ds-square).

Assume that $M^2(B)$ is not Euclidean. Then, due to [3], (10.2), there exist $u, v \in S$ such that $u \perp_B v$, but $u \not\perp_I v$, i.e., $||u + v|| \neq ||u - v||$. Let $k \in \mathbb{R}$ be such that $u + \tilde{v} \perp_B u - \tilde{v}$, where $\tilde{v} = k \cdot v$. Then $u \perp_B \tilde{v}$, $u + \tilde{v} \perp_B u - \tilde{v}$, but $||u|| \neq ||v||$ (if $|k| \neq 1$) or $||u + v|| \neq ||u - v||$ (if $|k| = 1$).

v) Let every dB-square be a BB-square. Thus, for all $u, v \in S$ with $u \perp_B v$ we have that $u+v \perp_B u-v$. This is only the case in the Euclidean plane; see [3], (10.1).

vi) Let finally every BB-square be a dB-square, i.e., if $u \perp_B v$ and $u+v \perp_B u-v$, then $||u|| = ||v||$. Analogously to *iv*), we assume that the plane is not Euclidean, and due to [3], (10.1), there exist $u, v \in S$ with $u \perp_B v$, but $u + v \not\perp_B u - v$. We choose $k \in \mathbb{R}$ such that $u + \tilde{v} \perp_B u - \tilde{v}$, where $\tilde{v} = k \cdot v$. Since $u \perp_B \tilde{v}$, $u + \tilde{v} \perp_B u - \tilde{v}$, and $||u|| = 1 \neq k = ||\tilde{v}||$, we have that u and $\tilde{v}$ form a BB-square, but not a dB-square. □

3.9 Outlook

Since there are many possibilities to extend this topic it is questionable whether it is possible to give a *complete* classification of quadrilaterals in normed planes.

Even only considering the definitions already given it is easy to add new types of rectangles and rhombi by using other types of orthogonality, for example Singer orthogonality [38], Roberts orthogonality [36], or Pythagorean orthogonality [23], just to list some important ones. Alonso and Benítez [1, 2] give some more definitions.

Further on, other Euclidean properties can be taken as definitions in normed planes and thus yield new types of rectangles/rhombi or even quadrilaterals that do not exist as an own type in Euclidean Geometry (as we have seen in the case of codises). For example, in Euclidean planes squares are exactly the quadrilaterals that posses concentric in- and circumcircles.

And finally there are many types of quadrilaterals that are so far not investigated, for example

- trapezoids (quadrilaterals with two parallel sides),
- kites (quadrilaterals that have two pairs of neighboured sides with equal length),
- cyclic quadrilaterals (quadrilaterals that are inscribed in a circle).

4 On conic sections

4.1 Introduction

It is surprising that not many results are known about the geometry of conics in arbitrary normed planes; see [42], [22], [39], [18], and [21]. In a natural way we continue here the investigations from [22]. It turns out that various well-known definitions of conics, equivalent in the Euclidean plane, do no longer coincide in normed planes, i.e., they can yield, in general, different types of curves. This is clarified in [22], and we use the most natural analogues of metrical definitions taken from there for our purpose (see below). Based on these definitions we present a collection of results describing conics in general normed planes and, particularly, also in polygonal normed planes. At least for the case of polygonal norms, we are also able to prove a theorem on bunches of Minkowskian ellipses and hyperbolas, which are pairwise Birkhoff orthogonal. Most results of this chapter are contained in [15].

4.2 Notation

For the definition of metric ellipses and hyperbolas we use similar notation as in [42] and [22]:

Definition 17 *Let $x, y \in M^2(B)$, and $c \in \mathbb{R}$ be such that $2c > \|x - y\|$. A* metric ellipse with foci *x* and *y* and of size *c is defined by*

$$E(x, y, c) := \{z \in M^2(B) : \|z - x\| + \|z - y\| = 2c\}.$$

Without loss of generality, we can consider the ellipse

$$E(x,c) := \{z \in M^2(B) \,:\, ||z-x|| + ||z+x|| = 2c\},$$

where $x \in C$. *In this case, the condition for* c *is reduced to* $c > 1$.

Due to [22] we have the following

Proposition 8 *Let* $E(x,c)$ *be a metric ellipse. Then*

$$E(x,c) = \{z \in M^2(B) \,:\, \exists r > 0 \,:\, B(z,r) \text{ touches } B(x,2c) \text{ from inside and contains -}x \text{ in its boundary}\}.$$

The boundary of $B(x,2c)$ *is called the* leading circle *of* $E(x,c)$.

This statement yields the possibility to visualise the shape of metric ellipses in an alternative way. We will use this more geometric definition later, in a proof.

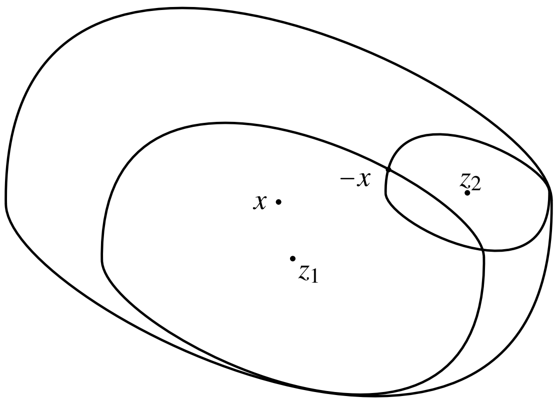

Figure 20 – z_1 and z_2 belong to an ellipse with foci x and $-x$.

In Figure 20, the points z_1 and z_2 are centres of circles that touch the large circle (which is centred at x and has radius $2c$) from inside and contain $-x$ in their boundaries. Thus, z_1 and z_2 belong to the ellipse $E(x,c)$.

Definition 18 *Let $x, y \in M^2(B)$ and $c \in \mathbb{R}$ be such that $0 < 2c < \|x - y\|$. A* metric hyperbola with foci x and y and of size $2c$ *is defined by*

$$H(x, y, c) := \{z \in M^2(B) : |\|z - x\| - \|z - y\|| = 2c\}.$$

Without loss of generality, we can consider

$$H(x, c) := \{z \in M^2(B) : |\|z - x\| - \|z + x\|| = 2c\},$$

where $x \in C$. In this case, the condition for c reduces to $0 < c < 1$.

In analogy to metric ellipses we have the following

Proposition 9 *Let $H(x, c)$ be a metric hyperbola. Then*

$$H(x, c) = \{z \in M^2(B) : \exists r > 0 : B(z, r) \textit{ touches } B(x, 2c) \textit{ from outside and contains -x in its boundary}\}.$$

The boundary of $B(x, 2c)$ is called the leading circle *of $H(x, c)$.*

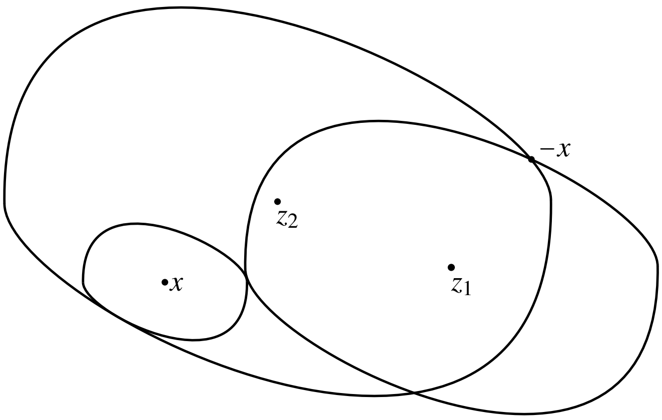

Figure 21 – z_1 and z_2 belong to a hyperbola with foci x and $-x$.

In Figure 21, the points z_1 and z_2 are centres of circles that touch the smallest circle (which is centred at x and has radius $2c$) from outside and contain $-x$ in their boundaries. Thus, z_1 and z_2 belong to the hyperbola $H(x, c)$.

Additionally, we define metric parabolas.

Definition 19 *Let $L \subset M^2(B)$ be a line and $x \in M^2(B)\backslash L$. A* metric parabola with focus x and leading line L *is defined by*

$$P(x, L) = \{z \in M^2(B) : ||z - x|| = \min\{||z - y||, y \in L\}\}.$$

Without loss of generality, we let $x \in S$ and L support B.

It is easy to see that the following statement holds.

Proposition 10 *Let $P(x, L)$ be a metric parabola. Then*

$$P(x, L) = \{z \in M^2(B) : \exists r > 0 : B(z, r) \textit{ touches } L \textit{ and contains } x \textit{ in its boundary}\}.$$

From now on, for the sake of simplicity, we sometimes write ellipse, hyperbola and parabola instead of metric ellipse, metric hyperbola and metric parabola.

4.3 On the shape of conics

In this section we will see that strict convexity (smoothness) of the unit disc is closely related to strict convexity (smoothness) of conic sections.

4.3.1 On metric ellipses

In [42] we find basic properties of metric ellipses.

Proposition 11 *Let $M^2(B)$ be a Minkowski plane. Then*

- *$E(x,c)$ is a centrally symmetric, closed convex curve for every $x \in C$ and $c > 1$, and*
- *$E(x,c)$ is strictly convex for every $x \in C$ and $c > 1$ if and only if B is strictly convex.*

Senlin Wu clarified the second property in an unpublished note.

Theorem 12 *Let $x \in S$, $c > 1$ and $y_1,\ y_2 \in E(x,c)$. Then the following two properties are equivalent:*

i) $[y_1, y_2] \subset E(x,c)$,
ii) $\left[\frac{x-y_1}{\|x-y_1\|}, \frac{x-y_2}{\|x-y_2\|}\right] \subset S$ and $\left[\frac{x+y_1}{\|x+y_1\|}, \frac{x+y_2}{\|x+y_2\|}\right] \subset S$.

This means that in general a segment in $E(x,c)$ is determined by two segments in the unit circle S. Senlin Wu also showed that, on the other hand, not any two segments in S yield a segment in $E(x,c)$.

Theorem 13 *Let $[a,b] \subset S$ and $[c,d] \subset S$, where a and c as well as b and d are linearly independent, $x \in S$ and $c' > 1$. Then $[y_1, y_2] \subset E(x,c')$, where $a = \frac{x-y_1}{\|x-y_1\|}$, $b = \frac{x-y_2}{\|x-y_2\|}$, $c = \frac{x+y_1}{\|x+y_1\|}$, and $d = \frac{x+y_2}{\|x+y_2\|}$, if and only if $|m| + |r| = |n| + |s|$, where $m, n, r, s > 0$ satisfy*

$$2x = ma + rc = nb + sd.$$

We introduce the notion of *corner points of curves* and add a theorem that yields information about the connection between smoothness of B and the smoothness of ellipses.

Definition 20 *Let C be a curve and $x \in C$. We call x a* corner point *of C, if C is non-smooth in x, i.e., there is no unique supporting line of C at x. The* supporting cone *of C at x is defined as the set of all the directions of supporting lines of C at x, and we call the two directions enclosing the cone* limit directions *of C at x.*

Theorem 14 *Let $M^2(B)$ be a Minkowski plane. Then $E(x,c)$ is smooth for every $x \in C$ and $c > 1$ if and only if B is smooth. More precisely, $z \in E(x,c)$ is a corner point of $E(x,c)$ if and only if $\frac{z-x}{||z-x||}$ or $\frac{z+x}{||z+x||}$ is a corner point of S.*

Proof: Let $x \in C$, $c > 1$, and $L = B(x, 2c)$ be the leading circle of $E(x,c)$.

According to Proposition 8 we consider a ball $B(z,r)$ that touches L from inside and contains $-x$ in its boundary. When the ball "moves" between L and x, its midpoint draws the ellipse $E(x,c)$; see Figure 20 on page 47. If we want to decide whether a point $z \in E(x,c)$ is a corner point of the ellipse, it suffices to study the movement of the ball (or rather its midpoint) in a neighbourhood of z.

From now on, y denotes the intersection point of the line (x,z) with L and thus a touching point of L with the ball $B(z,r)$. There are three possibilities for the locus of z.

- Case 1: Neither $\frac{z-x}{||z-x||}$ nor $\frac{z+x}{||z+x||}$ is a corner point of S. In this case, the ball moves locally between the unique supporting lines at x and y, and thus we have smoothness.

- Case 2: $\frac{z-x}{||z-x||}$ is a corner point of S. First we assume that $\frac{z+x}{||z+x||}$ is not a corner point of S. This means that y is a corner point of $B(z,r)$ (as well as of L), but -x is not. So the limit direction of the supporting lines of the ball in y moving into this position is different from the limit direction of the supporting lines at y when leaving this position, whereas the limit directions in $-x$ coincide. As in the first case, for every other position in a neighbourhood small enough, the change in direction of the movement of z happens exactly in this position, yielding that $E(x,c)$ is non-smooth.

- Case 3: $\frac{z+x}{||z+x||}$ is a corner point of S. Again we assume that $\frac{z-x}{||z-x||}$ is not a corner point of S. This means that $-x$ is a corner point of $B(z,r)$, but y is not. With the same argument as in case 2, we get that $E(x,c)$ is non-smooth in z.

It remains to state that if $\frac{z-x}{||z-x||}$ and $\frac{z+x}{||z+x||}$ are both corner points of S at the same time, then the line of midpoints changes its direction two times in one point. Since the ellipse is convex, this results again in a corner point. $\square$

Remark: The proof of Theorem 14 does not use the property that B is centrally symmetric. Thus the theorem holds also for *gauges*, i.e., for convex distance functions whose unit discs are not necessarily centrally symmetric.

Now we will have a closer look at normed planes with polygonal unit discs.

Definition 21 *A Minkowski plane is called* polygonal *if B is the convex hull of finitely many points.*

For information on conics in a special polygonal plane, namely the rectangular plane, we refer to [26].

Theorem 14 yields the following corollary.

Corollary 4 *Let $M^2(B)$ be a polygonal Minkowski plane. Then for all $x \in S$ and $c > 1$ the ellipse $E(x,c)$ is a polygon. Moreover, the possible directions of the sides of $E(x,c)$ do not depend on x or c, but only on the shape of B. More precisely, let s be one side of the polygon $E(x,c)$. Then there exist two sides of B such that the lines containing these sides intersect in a point u, where $(o,u) \parallel s$.*

Proof: Both statements follow from the main idea of the proof. We use the same notions and consider the disc $B(z,r)$ that moves between $-x$ and L. If B is polygonal, then the supporting lines of $B(z,r)$ at $-x$ and y do not change, unless one of these two points is a corner point. Let these two supporting lines intersect in a point v. Then z moves (locally) on the line (z,v). If on the other hand the two supporting lines are parallel, then z moves on a line parallel to them.

In particular, z moves on a straight line between each two of the corner points of the ellipse, and the direction of this straight line (=side of $E(x,c)$) does not depend on the position of x or the size of the ellipse, but only on the (local) properties of the supporting lines of $B(z,r)$ in $-x$ and y. □

4.3.2 On metric hyperbolas

Due to [22] the following holds.

Proposition 12 *Let $M^2(B)$ be a Minkowski plane. Its unit ball B is strictly convex if and only if for every $x \in C$ and $0 < c < 1$ the hyperbola $H(x,c)$ consists of two simple curves, called* branches*, where each of them is intersected by any line parallel to (o,x) in exactly one point.*

We add the following

Proposition 13 *Let $M^2(B)$ be a Minkowski plane that is not strictly convex, and let $[u,v] \subset S$ be a straight segment in the boundary of B. Let $x \in M^2(B)$ be such that (o,x) is not parallel to (u,v). Then there exists a value $0 < c < 1$ such that $H(x,c)$ is not the union of two simple curves.*

Proof: We consider the leading circle $L = B(x, 2c)$, where c is chosen such that $x+v \in [x+u, -x]$; see Figure 22. Let $z \in M^2(B)$ be the intersection point of $(x, x+v)$ and $(-x, -x+u)$. Then every point of the cone $\{z + s \cdot u + t \cdot v,\ s,t \geq 0\}$ has equal distance to $-x$ and $x + v$. Hence it belongs to $H(x,c)$. □

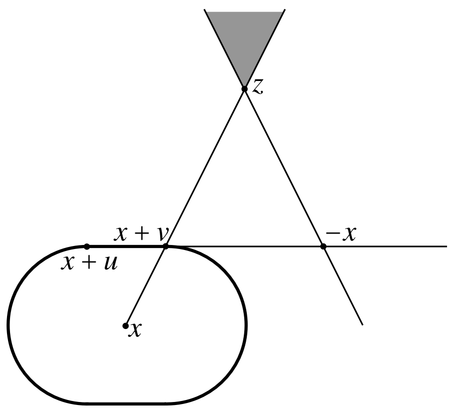

Figure 22 – Every point of the shaded cone belongs to the hyperbola.

Concerning corner points, we have a result for hyperbolas which is similar like for ellipses.

Theorem 15 *Let $M^2(B)$ be a Minkowski plane. Let $x \in C$ and $0 < c < 1$ be such that $H(x,c)$ consists of two simple curves. If B is smooth, then the branches of $H(x,c)$ are smooth. More precisely, if $z \in H(x,c)$ is a corner point of $H(x,c)$, then $\frac{z-x}{||z-x||}$ or $\frac{z+x}{||z+x||}$ is a corner point of S.*

Proof: The proof works in the same way as the one for ellipses (Theorem 14). With analogous arguments we have that, for every corner point z of $H(x,c)$, $\frac{z-x}{||z-x||}$ or $\frac{z+x}{||z+x||}$ is a corner point of S. □

<u>Remark:</u> Note that, in general, the other direction is not true. Assume that $\frac{z-x}{||z-x||}$ and $\frac{z+x}{||z+x||}$ are both corner points of S. In this case, again we have two changes in direction. But as the branches of $H(x,c)$ do not have to be convex, they may cancel out each other; see Figure 23 for an example.

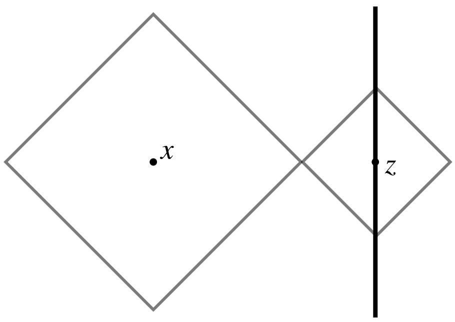

Figure 23 – The hyperbola (thick line) has a "double corner point" in z.

4.3.3 On metric parabolas

Theorem 16 *Let L be a line supporting B, and let $x \in S \setminus L$. Then $P(x,L)$ is a simple, convex curve.*

Proof: Ghandehari [18] showed that the area bounded by the parabola is convex.

It remains to prove that $P(x, L)$ is a simple curve. For this reason we show that the ray $\{x + t \cdot u,\ t > 0\}$ starting in x with direction $u \in S$ intersects $P(x, L)$ in at most one point. Assume that there are different values $s, t > 0$ such that $x + t \cdot u = y_1 \in P(x, L)$ and $x + s \cdot u = y_2 \in P(x, L)$. Obviously, this is not possible for $(o, u) \parallel L$. Let $z = (x, x + u) \cap L$, and let w_1, w_2 be from L with minimal distance to y_1, y_2, respectively; see Figure 24. Then similarity of triangles yields

$$\frac{||y_1 - z||}{||y_1 - w_1||} = \frac{||y_2 - z||}{||y_2 - w_2||} \iff \frac{||y_1 - z||}{||y_1 - x||} = \frac{||y_2 - z||}{||y_2 - x||}$$
$$\iff (||y_1 - x|| + ||x - z||)\,||y_2 - x|| = (||y_2 - x|| + ||x - z||)||y_1 - x||$$
$$\iff ||x - z|| \cdot ||y_2 - x|| = ||x - z|| \cdot ||y_1 - x|| \iff ||x - z|| \cdot ||y_2 - y_1|| = 0.$$

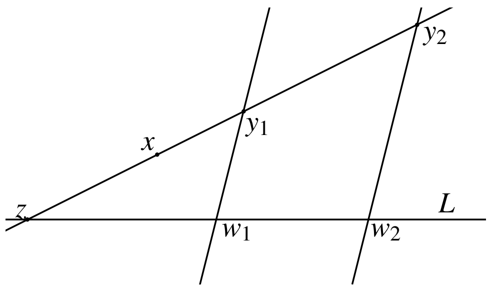

Figure 24 – The lines intersecting L are Birkhoff orthogonal to L.

But since $x \notin L$ and $y_1 \neq y_2$, this is a contradiction. $\square$

It is not surprising that we have a result on corner points of a parabola, and that the proof works in the same way as for ellipses and hyperbolas.

Theorem 17 *Let L be a line supporting B, and let $x \in S \backslash L$. The unit disc B is smooth if and only if $P(x, L)$ is smooth. More precisely, $z \in P(x, L)$ is a corner point of $P(x, L)$ if and only if $\frac{z-x}{||z-x||}$ is a corner point of S.*

With the two results above we are able to prove the following theorem that is well known for the Euclidean subcase.

Theorem 18 *Let L be a line supporting B, and let $x \in S \backslash L$. Let $U = S \cap L$. Then*

i) the ray $\{x + tu,\ t > 0\}$ does not intersect $P(x, L)$ for all $u \in U$,
ii) the ray $\{x - tu,\ t > 0\}$ does intersect $P(x, L)$ for all $u \notin U$.

Proof: First we assume that $M^2(B)$ is a polygonal Minkowski plane and that $U = [u, v]$, where $u \neq v$. Let $w_1 \in S$ be the corner point of B such that v lies between u and w_1 (with respect to the boundary of B). Let the line passing through x with direction $w_1 - v$ intersect L in w_1'. Then each circle centered at $m(r) = w_1' - r \cdot v$, where $r \geq r_0 = \frac{||w_1' - x||}{||w_1 - v||}$, touches L and contains x in its boundary; see Figure 25. The equality $r = r_0$ yields that $m(r) + r \cdot v = w_1'$ and $m(r) + r \cdot w_1 = x$. For $r > r_0$, we still have $m(r) + r \cdot v = w_1'$. In addition we have $x \in [w_1', m + r \cdot w_1]$. Thus, $m(r) \in P(x, L)$ for all $r \geq r_0$.

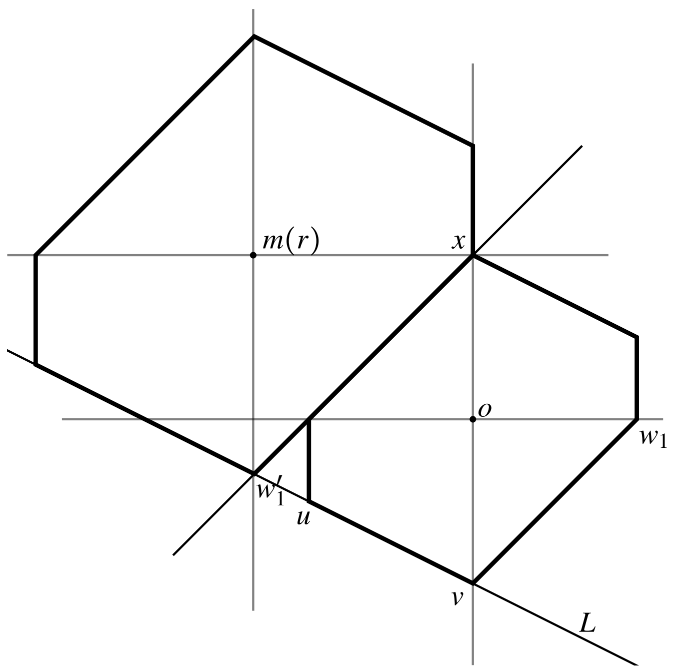

Figure 25 – Theorem 18 for the polygonal case.

Analogously we can define w_2 and w_2', and we get that $w_2' - r \cdot u \in P(x, L)$ for all $r \geq \frac{||w_2'-x||}{||w_2-u||}$. Thus, $x - s \cdot u - t \cdot v \notin P(x, L)$ for all $s, t > 0$. Clearly, every other ray starting in x intersects $P(x, L)$.

For $u = v$, let w_1 and w_2 be the corner points of B such that u lies between w_1 and w_2. Then the same arguments as above yield that the ray $x - t \cdot u$ is the only one that does not intersect the parabola.

The non-polygonal case follows by continuity. Let $n \in N$, and let n points be equally distributed in the boundary of a smooth unit disc B. The theorem holds for the convex hull of these points. With $n \to \infty$, the convex hull converges to B. □

Another interesting statement is given by

Theorem 19 *Let L be a line supporting B, and let $x \in S \backslash L$. Let $y, z \in P(x, L)$ be such that $x \in (y, z)$, where neither y nor z is a corner point of $P(x, L)$. Then the tangent lines of $P(x, L)$ in y and z intersect in L.*

Proof: Again we discuss the case where $M^2(B)$ is a polygonal plane and conclude the general situation by continuity arguments.

As y is not a corner point of $P(x, L)$, we have that x is not a corner point of the disc $B(y, ||y - x||)$. Let s_y be the (unique) line that supports this disc at x. From the former theorem we know that s_y intersects L. If it were parallel to L, then the line through y and x would not intersect the parabola for a second time. We call the intersection point p_y.

Consider now the disc that moves between L and x. Its midpoint "draws" the parabola. The tangent line t_y of $P(x, L)$ at y is determined by the movement of the center of the disc in a neighbourhood of y. As the disc moves locally between L and s_y, every point of B moves towards the intersection point of s_y and L; see Figure 26. In particular, $p_y \in t_y$.

On the other hand, with the same arguments there is a unique line s_z that supports $B(z, ||z - x||)$ at x. Again we have that s_z intersects L, and every point of the disc, that moves between x and L in a neighbourhood of z, moves towards this intersection point. We call it p_z. In particular, the tangent line t_z of $P(x, L)$ at z contains p_z.

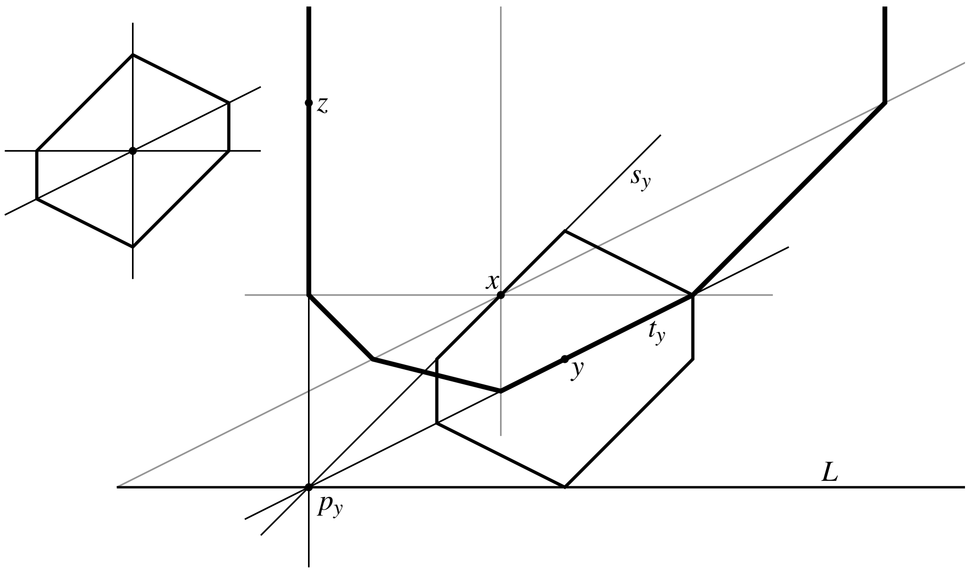

Figure 26 – Notation in the proof of Theorem 19.

But since x lies on the line through y and z, we have that y and z lie on opposite and therefore parallel sides of the (corresponding) leading circle. Thus $s_y \parallel s_z$. Since both of them contain x, they are equal. It follows that $p_y = p_z$, hence t_y and t_z intersect in L. $\square$

4.4 Intersecting ellipses and hyperbolas

For the Euclidean plane, the following statement is well known; see, e.g., [4].

Theorem *Let $E(x, c),\ c > 1$, be an ellipse and $H(x, d),\ 0 < d < 1$, a hyperbola with arbitrary, but identical foci x and $-x$. Let t_e and t_h be tangent lines at a common point of E and H. Then $t_e \perp t_h$.*

A similar, but slightly weaker theorem holds for polygonal Minkowski planes.

Theorem 20 *Let $M^2(B)$ be a polygonal Minkowski plane, and $x \in S$ be arbitrary. Let $E(x, c),\ c > 1$, be an ellipse and $H(x, d),\ 0 < d < 1$, a hyperbola that consists* *of two simple curves. Let t_e and t_h be tangent lines at a common point of E and H. If c is large enough, then $t_h \perp_B t_e$.*

Later on we will see what "large enough" means in this context, and what happens in the case that c is small.

To prove the theorem, we need some additional notation. If $M^2(B)$ is a polygonal plane, then there are only finitely many lines each passing through the origin and a corner point of B. We call the lines parallel to these and passing through x and $-x$ *corner lines*. These corner lines divide the plane into (bounded and unbounded) cells; see Figure 27. Any corner point of any metric conic with foci x and $-x$ lies on a corner line; within the cells there are only straight segments. We will have now a closer look at the unbounded cells.

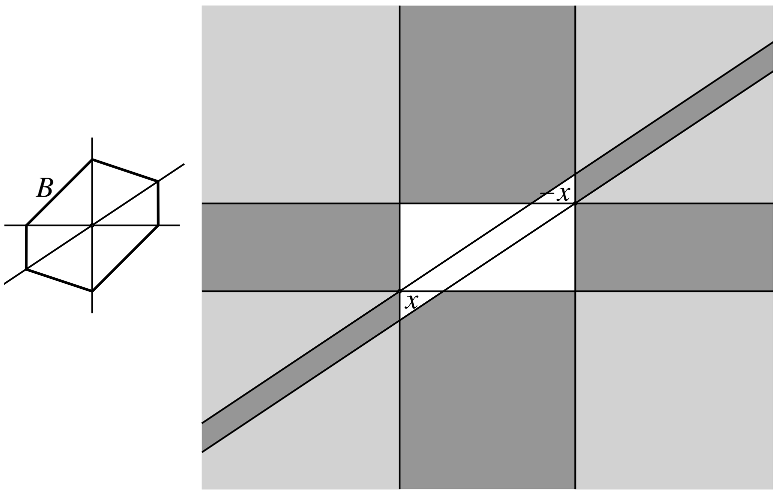

Figure 27 – Cones (light grey) and strips (dark grey).

It is easy to see that the unbounded cells are either strips, bounded by two parallel corner lines, or cones, bounded by two corner lines, corresponding to two adjacent corner points of B. In this sense, each strip is corresponding to a corner point of B, and each cone is corresponding to one side of the polygon B.

The following proposition yields information about the behaviour of ellipses and hyperbolas inside the cones.

Proposition 14 *Let p_1 and p_2 be two points that lie in the boundary of one of the cones, but on different corner lines, such that the line $[p_1, p_2]$ is parallel to the corresponding side of B; see Figure 28. Denote by q the vertex of the cone. Then we have*

i) $||p - x|| + ||p + x|| = ||p_1 - x|| + ||p_1 + x||$ *for all* $p \in [p_1, p_2]$*, and*
ii) $|\,||p - x|| - ||p + x||\,| = |\,||q - x|| - ||q + x||\,|$ *for all* $p \in [p_1, p_2]$.

In other words, if one point of the segment $[p_1, p_2]$ belongs to an ellipse with foci x and $-x$, then any point of the segment does. If one point of the cone belongs to a hyperbola with foci x and $-x$, then any point of the cone does.

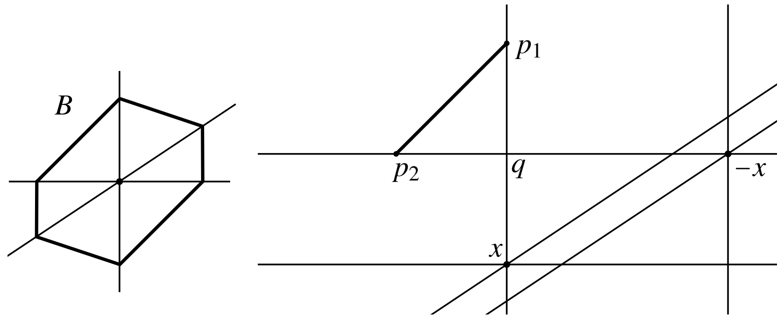

Figure 28 – (p_1, p_2) is parallel to the corresponding side of B.

Proof: Obviously, $[p_1, p_2] \subset B(x, ||p_1 - x||) \cap B(-x, ||p_2 + x||)$. Thus $||p - x|| = ||p_1 - x||$ and $||p + x|| = ||p_2 + x|| = ||p_1 + x||$. In particular, *i)* holds.

Additionally, $[p_1, p_2] \subset B(q, ||p_1 - q||)$, i.e., $||p - q|| = ||p_1 - q||$ for all $p \in [p_1, p_2]$. Thus we have

$$\begin{aligned} |\,||p - x|| - ||p + x||\,| &= |\,||p_1 - x|| - ||p_2 + x||\,| \\ &= |(||p_1 - q|| + ||q - x||) - (||p_2 - q|| + ||q + x||)| \\ &= |\,||q - x|| - ||q + x||\,|, \end{aligned}$$

proving *ii)*. $\square$

Now we study the behaviour of hyperbolas within the strips.

Proposition 15 *Let $u \in S$ be a corner point of B. There are two strips with sides parallel to the line (o, u). We consider the strip for which with every point p also the ray $R = \{p + t \cdot u, t \in \mathbb{R}\}$ lies within that strip. If $p \in H(x, c)$ for any $0 < c < 1$, then $R \subseteq H(x, c)$. In other words: Whenever a hyperbola enters a strip, it stays inside.*

Proof: Let $p \in H(x, c)$ be any point of the hyperbola that lies within the strip, and q be any point of the ray R. We define a and b as the points that lie on the cornerline that passes through x, such that $||x - p|| = ||x - a||$ and $||x - q|| = ||x - b||$. Analogously, c and d are defined as the points lying on the corner line passing through $-x$, such that $||-x - p|| = ||-x - c||$ and $||-x - q|| = ||-x - d||$; see Figure 29.

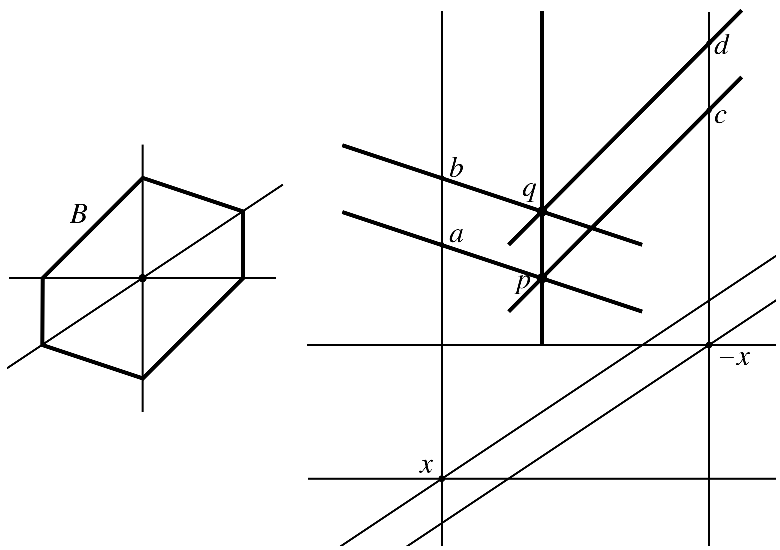

Figure 29 – $p \in H(x, c) \iff q \in H(x, c)$.

We note that $||p - q|| = ||a - b|| = ||c - d||$. Then we have

$$
\begin{aligned}
\|x-p\| - \|-x-p\| &= \|x-a\| - \|-x-c\| \\
&= \|x-a\| + \|a-b\| - \|-x-c\| - \|c-d\| \\
&= \|x-b\| - \|-x-d\| \\
&= \|x-q\| - \|-x-q\| .
\end{aligned}
$$

In particular, the absolute values of the first and the last difference are equal, and thus $q \in H(x,c)$. □

Since ellipses are convex curves, we have that the slope of the ellipse within a strip is somewhere between the slopes in the adjacent cones, and thus somewhere between the slopes of the two sides of B that are adjacent to u.

Finally, we are able to prove Theorem 20.

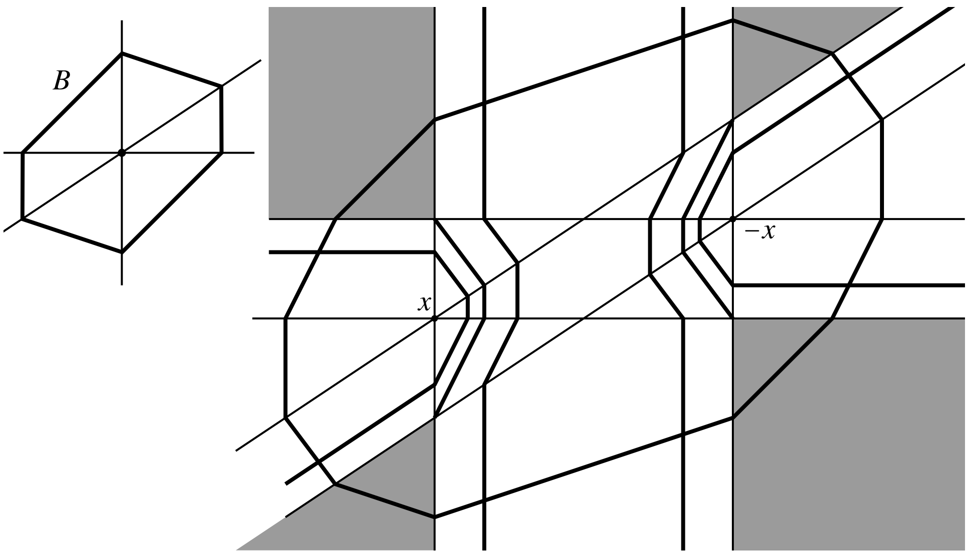

Figure 30 – The hyperbolas intersect the ellipse orthogonally in the sense of Birkhoff.

Proof of Theorem 20: Let $c > 1$ be such that every intersection point of $E(x,c)$ with $H(x,d)$ lies within one of the unbounded cells. Since $H(x,d)$ does not contain a cone, all these points have to lie within strips. Thus we have that $t_h \parallel (o,u)$. By definition, (o,u) is Birkhoff orthogonal to the two sides of B adjacent to u as well as to any line that has a slope that lies between the slopes of these two sides. Thus, $t_h \perp_B t_e$. $\square$

<u>Remark:</u> The statement of the theorem can be extended to arbitrary hyperbolas: Let $E(x,c)$ be an ellipse and let $H(x,d)$ be a hyperbola that contains a cone, and consider this cone as a union of rays. Since the side of the ellipse within this cone is parallel to the appropriate side of B, every ray is Birkhoff orthogonal to this side of $E(x,c)$; see Figure 30.

Theorem 20 does not hold for every $c > 1$ in arbitrary Minkowski planes; see the counterexample in Figure 31.

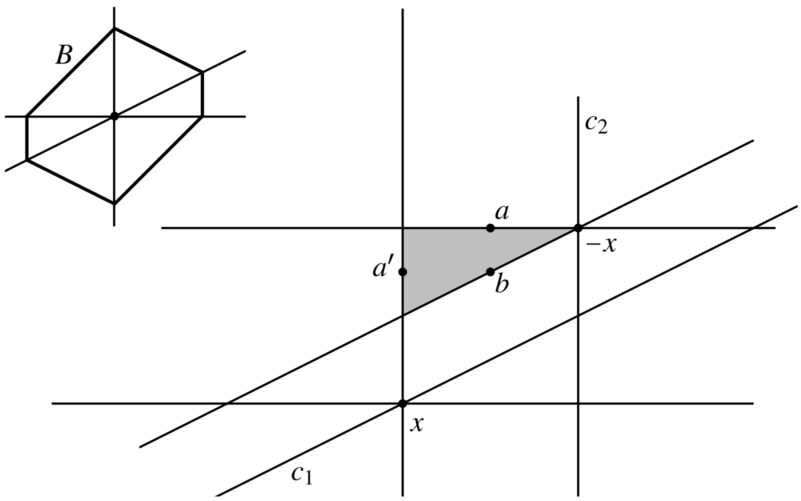

Figure 31 – Counterexample if c is not large enough.

We consider the ellipse $E = E(x, 3.5)$. Within the shaded cell the boundary of E is a line that is parallel to the cornerline c_1. For example, a and a' belong to E since $\|x-a\| + \|x-a'\| = \|-x-a\| + \|-x-a'\| = 3.5$. This means that only lines parallel to the vertical cornerline c_2 are Birkhoff orthogonal to E (within the shaded cell).

However, we have that $||-x-a|| = ||-x-b|| = 1$, but obviously $||x-a|| > ||x-b||$ (more precisely, $||x-a|| = 2$ and $||x-b|| = 2.5$). As a consequence, the hyperbola H that passes through a does not pass through b, and thus it does not move parallel to c_2 within the shaded cell. Hence, there the ellipse E and the hyperbola H do not intersect orthogonally in the sense of Birkhoff.

On the other hand, it is possible that the statement holds for every $c > 1$ in certain normed planes; see Figure 32 for an example.

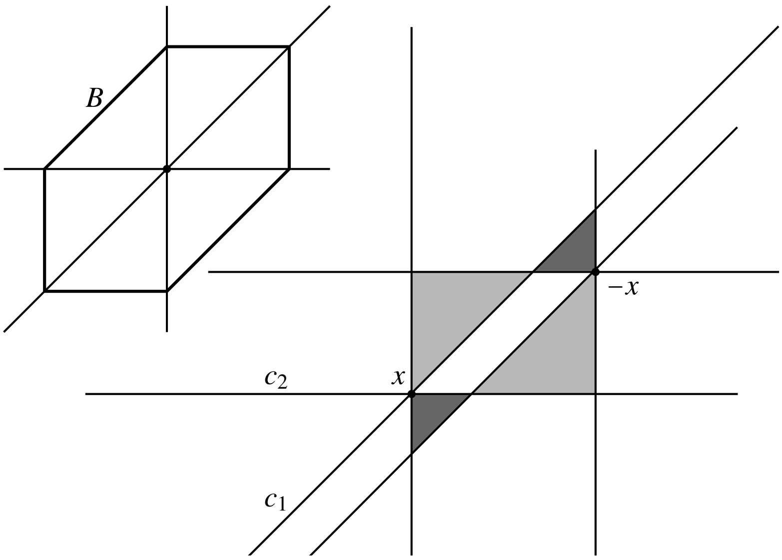

Figure 32 – Also for arbitrary $c > 1$, Theorem 20 can be true.

Let E be an ellipse that passes through the bounded (shaded) cells (which is the case $2c < 2.5$). Similar to the counterexample above, E is parallel to cornerlines within these cells, namely parallel to c_1 in the brighter cells and parallel to c_2 in the darker ones. But since the plane is Radon, the corner lines are parallel to the sides of B, and thus the statement of Theorem 20 holds.

It remains to figure out whether or not the theorem holds in non-polygonal planes. Contrary to Theorems 18 and 19 it is not possible simply to use a "continuity argument": Let B be a non-polygonal convex body, and let B_n be an approximation of B by a polygonal body with n cornerpoints and such that $B_n \to B$ for $n \to \infty$. Then some neighbourly corner lines of B_n are nearly parallel for large n and thus intersect "far away" from the origin. This means that Theorem 20 holds only for very large values of c. With increasing extent, ellipses converge to circles, since - compared to the size of the ellipse - the foci x and $-x$ are almost the same point. Thus we can give only the following result for non-polygonal planes.

Corollary 5 *Let $M^2(B)$ be any Minkowski plane and let $H(x,d),\ 0 < d < 1,$ be a hyperbola that consists of two simple curves. Then each of these two curves has two asymptotes that intersect (a very large copy of) the unit disk B orthogonally in the sense of Birkhoff. In particular, if B is smooth, then the asymptotes pass through the origin.*

In conclusion we have that for the case of non-polygonal planes it is essential to know what happens inide the bounded cells. Though we can describe both the directions of an ellipse and of a hyperbola within a certain cell, we are not able to give a general formula.

4.5 Outlook

The topic of conic sections in Minkowski planes can be extended in various ways. First of all, one can look for alternative definitions in the Euclidean plane and consider the appropriate sets in normed planes. While the definition of an ellipse (or hyperbola) by one focus and the leading circle yields the same objects as the definition by two foci (see Proposition 8 on page 47), Horváth and Martini [22] gave another definition for ellipses in normed planes.

Definition 22 *Let l be a straight line, x a point, and $\gamma = \frac{a}{c}$ a ratio larger than 1. The locus of points $z \in X$, for which there is a positive ε such that the boundary of the disk $z + \varepsilon B$ contains x and the disk $z + \gamma(\varepsilon B)$ touches the line l, will be called the* ellipse defined by its leading line and its focus *x.*

Figure 33 shows an example of ellipses defined by a focus and a leading line that are obviously not ellipses defined by two foci (since they are not symmetric).

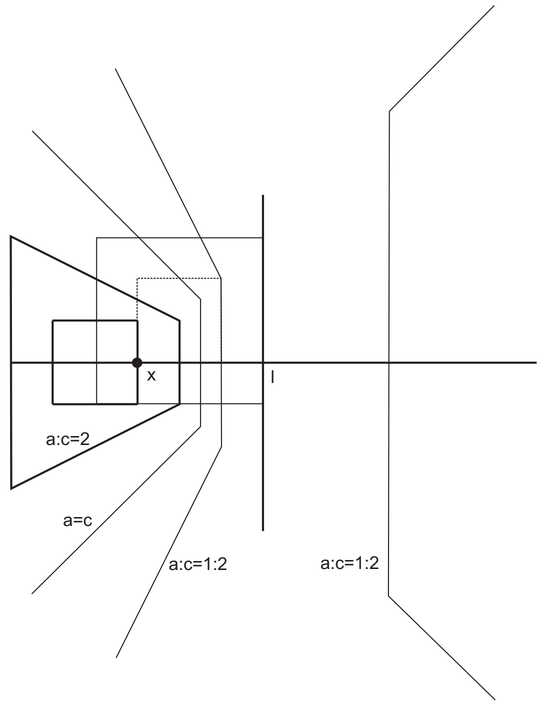

Figure 33 – Example for ellipses defined by a leading line and a focus that are not ellipses defined by two foci.

Horváth and Martini gave, on the other hand, also an example for an ellipse defined by two foci that is not an ellipse defined by a focus and a leading line. Hence this definition yields a different type of curves, that are less Euclidean-like (since they lack symmetry).

Further on, conics can be considered as what they are in the Euclidean plane originally: conic sections. More precisely, consider the 3-dimensional Euclidean space with coordinates x, y and z, and let B be a symmetric convex bounded closed set lying in the plane $z = 1$ with center $(0,0,1)$ and such that this central point lies in the relative interior of B, i.e., there is an open neighbourhood N of $(0,0,1)$ such that $N \cap \mathrm{aff} B \subset B$. Thus, the plane $z = 1$ can be identified with $M^2(B)$. Then we can define conics as the intersections between an arbitrary plane (in the 3-dimensional space) with the boundary of the cone $\{k \cdot B,\ k \in \mathbb{R}\}$.

With this new definition in mind we have another look at Figure 33. Let $B :=$ conv$\{(1,1,1),(-1,1,1),(-1,-1,1),(1,-1,1)\} \subset \{z = 1\} \subset \mathbb{R}^3$, i.e., B is a square, and consider a plane $E := ay + z = d$, where $|a| < 1$ and $d \in \mathbb{R}$. Then E intersects the boundary of the cone $\{k \cdot B,\ k \in \mathbb{R}\}$ in a single curve (since a is small enough) that is symmetric to the plane $x = 0$ and has two sides that are parallel to the x-axis (since x is a free variable). Summarized, the curve looks exactly like the ellipses in Figure 33, that where defined in the plane $M^2(B)$ (with the same B as above) by a focus and a leading line. Thus, there might exist a connection between these two definitions.

Another property of ellipses in Euclidean Geometry is used in physics and astronomy. Consider one focus of an ellipse as a light source. Then any light ray starting from this focus is reflected by the boundary of the ellipse into the other focus; see [37] for a proof. Ghandehari [18] deals with this problem in normed planes. Considering our first chapter, the following question arises.

Question *Is it possible to define an angular measure in a Minkowski plane such that for each ellipse $E(x,c)$ the following property holds: Let $y \in E(x,c)$, and let t be a line supporting $E(x,c)$ in y. Then (x,y) and $(-x,y)$ form the same angle with the line t; see Figure 34.*

A similar question can be asked for parabolas, since in the Euclidean plane light rays that are orthogonal to the leading line L of a parabola $P(x,L)$ are reflected into the focus x.

Obviously, straight lines and corner points in conics cause non-trivial zero angles, which leads to the conjecture that in polygonal planes, where B has only few corner points, no such an angular measure exists, since the zero angles are too large (in terms of arc length). Unfortunately, up to now we do not know enough about conics in other planes, especially in planes that are smooth and strictly convex.

There is a lot of more results on conic sections in the Euclidean plane that may be worth to be discussed. We only give two examples.

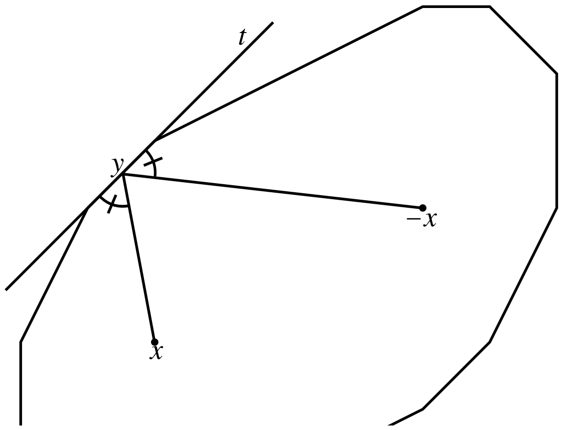

Figure 34 – Does this property define an angular measure?

Theorem 21 ***(Ivory's theorem)*** *Let E_1, E_2, H_1 and H_2 be two ellipses and two hyperbolas with the same foci, and let their intersection points form a curvilinear quadrangle Q. Then the diagonals of q have the same length.*

This statement does not hold in general Minkowski planes. In fact, it is very easy to give a counterexample.

Theorem 22 *Let x, y, z be three points that do not lie on one line. Let E be an ellipse with foci x and y and C a circle with center z such that E and C touch each other. Then the touching point is the Fermat-Torricelli point of x, y, and z, i.e., it is the point with minimal distance sum to the three given points x, y, z.*

This theorem gives a connection of our topic to the famous Fermat-Toricelli problem: to find the set of points that minimise the distance to n given points. See [29], [9] and [33] for results on this topic in normed planes.

List of Figures

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Theses for the dissertation

"Metrical Problems in Minkowski Geometry",

submitted by Dipl.-Math. Andreas Fankhänel,
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1. The submitted dissertation contains new metrical results on the geometry of 2-dimensional normed linear spaces (or normed planes).

2. We define different angular measures and examine in what normed planes such measures exist.

3. There exists an i-measure in a normed plane $M^2(B)$ with unit disc B, i.e., an angular measure such that in any isosceles triangle the base angles are of equal value if and only if $M^2(B)$ is the Euclidean plane. In that case, μ is the usual Euclidean angular measure.

4. There exists a B-measure in $M^2(B)$, i.e., an angular measure such that the angle between two Birkhoff orthogonal vectors has value $\frac{\pi}{2}$, if and only if the unit circle of $M^2(B)$ contains Radon arcs.

5. There exists a T-measure in $M^2(B)$, i.e., an angular measure such that the angle between two vectors that are isosceles orthogonal has value $\frac{\pi}{2}$, if and only if the plane is Euclidean. In that case, the measure is a B-measure.

6. We begin to create a classification of convex quadrilaterals and give different definitions for rectangles and rhombi, based on connections between sides and/or diagonals. Also we define a new type of quadrilaterals, called codises, and squares.

7. We characterise types of normed planes where different kinds of rectangles (rhombi) coincide.

8. A plane is a Radon plane if and only if every codis is a rectangle (or vice versa), or if every parallelogram, that posseses an incircle, is a rhombus (or vice versa).

9. We give metrical definitions of conic sections, more precisely of metric ellipses, metric hyperbolas, and metric parabolas.

10. We prove that corner points of the unit disc B cause corner points in conic sections: If H is a metric hyperbola and $z \in H$ a point such that $\frac{z-x}{||z-x||}$ or $\frac{z+x}{||z+x||}$ is a corner point of S, then z is a corner point of H. For metric ellipses also the other direction is true. If P is a metric parabola and $z \in P$ a point, then z is a corner point of P if and only if $\frac{z-x}{||z-x||}$ is a corner point of S.

11. For any $y, z \in P(x, L)$, such that y and z are no longer corner points of P, and $x \in [y, z]$ we have that $t_y \cap t_z \in L$, where t_y and t_z are lines that support the parabola at y and z, respectively.

12. In polygonal planes, confocal hyperbolas and ellipses intersect orthogonally in the sense of Birkhoff whenever the respective ellipse is large enough.

Lebenslauf

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Erklärung

Ich erkläre an Eides Statt, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Chemnitz, den 30.01.2012 Andreas Fankhänel